International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2013): 4.438

Zeros of Polar Derivative with Respect to A Real Number

G. L. Reddy[1], K. Anoosha[2], K. Sravani[3]

[1, 2, 3] School of Mathematics and Statistics, University of Hyderabad - 500046, India

Abstract: ***In this paper we obtain the size of the disc in which the zeros of polar derivatives of polynomial of degree n, with real coefficients, with respect to a real α lie.***

Keywords: zeros, polar derivatives, polynomials, real number

1. Introduction

To estimate the zeros of a polynomial is a long standing problem. It is an interesting area of research for many engineers as well as mathematicians and many results on the topic are available in the literature.

Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n. Then Polar Derivative of the polynomial P(z) with respect to α, where α can be real or complex number, is defined as
$D_\alpha P(z)=n\,P(z)+(\alpha-z)P'(z)$.

It is a polynomial of degree at most n-1.The polynomial $D_\alpha P(z)$ generalizes the ordinary derivative, in the sense that $\lim_{n\to\infty} D_\alpha P(z)/\alpha = P'(z)$.

In this paper we prove the following results.

Theorem (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$
$ka_n \leq a_{n-1} \leq \dots \leq a_0 < 0$
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-1.
Then the polar derivative of P(z) with respect to α has at most (n-1) zeros and they lie in
$|z+k-1| \leq k$.

Theorem (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$
$ka_n \leq a_{n-1} \leq \dots \leq a_0$
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-1.
Then the polar derivative of P(z) with respect to α has at most (n-1) zeros and they lie in
$|z+k-1| \leq |a_{n-1} + \alpha n a_n|^{-1}\{-k(a_{n-1} + \alpha n a_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.

Corollary (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that
$a_n \leq a_{n-1} \leq \dots \leq a_0$
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-1.
Then the polar derivative of P(z) with respect to α has at most (n-1) zeros and they lie in
$|z| \leq |a_{n-1} + \alpha n a_n|^{-1}\{-(a_{n-1} + \alpha n a_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.

Corollary (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$
$ka_n \leq a_{n-1} \leq \dots \leq a_0$
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .n-1.
Then the polar derivative of P(z) with respect to $\alpha \neq -a_{n-1}/na_n$

has exactly (n-1) zeros and they lie in
$|z+k-1| \leq |a_{n-1} + \alpha n a_n|^{-1}\{-k(a_{n-1} + \alpha n a_n) + na_0 + \alpha n a_1 + |na_0 + \alpha n a_1|\}$.

Remark :(1) By taking k=1 in Theorem (2), it reduces to corollary (1) .
(2) As $\alpha \neq -a_{n-1}/na_n$, $D_\alpha P(z)$ is surely of (n-1)th degree so it has exactly (n-1) zeros.

Theorem (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that for some $k \geq 1$
$ka_m \leq a_{m-1} \leq \dots \leq a_0$ where m= 0, 1, 2……n
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .m-1.
Then the polar derivative of P(z) with respect to α such that
$\alpha = -a_{n-1}/na_n = -2a_{n-2}/(n-1)a_{n-1} = \dots = -(n-m-1)a_{m+1}/(m+2)a_{m+2}$

has exactly m roots and they lie in
$|z+k-1| \leq |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-k(n-m)a_m - k\,\alpha(m+1)a_{m+1} + na_0 + \alpha n a_1 + |na_0 + \alpha n a_1|\}$.

Corollary (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n with real coefficients such that
$a_m \leq a_{m-1} \leq \dots \leq a_0$ where m= 0, 1, 2……n
and $ia_i \leq (i-1)a_{i-1}$ i= 0, 1, 2, . . .m-1.
Then the polar derivative of P(z) with respect to α such that
$\alpha = -a_{n-1}/na_n = -2a_{n-2}/(n-1)a_{n-1} = \dots = -(n-m-1)a_{m+1}/(m+2)a_{m+2}$

has exactly m roots and they lie in
$|z+k-1| \leq |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\{-(n-m)a_m - \alpha(m+1)a_{m+1} + na_0 + \alpha n a_1 + |na_0 + \alpha n a_1|\}$.
Remark (2): By taking k=1 in Theorem (3), it reduces to corollary (3).

2. Proof of Theorem 1

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ be a polynomial of degree n.
Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then
$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \dots$

$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$Q(z) = -[a_{n-1} + \alpha n a_n]z^n + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

$= -[a_{n-1} + \alpha n a_n][z+k-1]z^{n-1} + [k(a_{n-1} + \alpha n a_n) - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-n} < 1$ for $i = 1, 2, 3, \ldots n-1$

Further

$|Q(z)| \ge |a_{n-1} + \alpha n a_n||z+k-1||z|^{n-1} - \{|k(a_{n-1} + \alpha n a_n) - 2a_{n-2} - \alpha(n-1)a_{n-1}||z|^{n-1}$

$+ \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}||z|^{m+1}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||z|^m$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}||z|^{m-1}$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2||z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1||z|$

$+|na_0 + \alpha a_1|\}$.

$\ge |a_{n-1} + \alpha n a_n||z|^{n-1}[|z+k-1| - |a_{n-1} + \alpha n a_n|^{-1}\{|k(a_{n-1} + \alpha n a_n) - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}||z|^{-1} + \ldots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}||z|^{-(n-m-2)}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||z|^{-(n-m-1)}$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}||z|^{-(n-m)} + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2||z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2$

$- na_0 - \alpha a_1||z|^{-(n-2)} + |na_0 + \alpha a_1||z|^{-(n-1)}\}]$.

$\ge |a_{n-1} + \alpha n a_n||z|^{n-1}[|z+k-1| - |a_{n-1} + \alpha n a_n|^{-1}\{|k(a_{n-1} + \alpha n a_n) - 2a_{n-2}$

$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$

$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| + |na_0 + \alpha a_1|\}]$.

$\ge |a_{n-1} + \alpha n a_n||z|^{n-1}[|z+k-1| - |a_{n-1} + \alpha n a_n|^{-1}\{-k(a_{n-1} + \alpha n a_n) + 2a_{n-2}$

$+\alpha(n-1)a_{n-1} - 2a_{n-2} - \alpha(n-1)a_{n-1} + 3a_{n-3} + \alpha(n-2)a_{n-2} + \ldots$

$-(n-m-1)a_{m+1} - \alpha(m+2)a_{m+2} + (n-m)a_m + \alpha(m+1)a_{m+1}$

$-(n-m)a_m - \alpha(m+1)a_{m+1} + (n-m+1)a_{m-1} + \alpha m a_m$

$-(n-m+1)a_{m-1} - \alpha m a_m + (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} + \ldots$

$-(n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2 - (n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1$

$+|na_0 + \alpha a_1|\}]$.

$\ge |a_{n-1} + \alpha n a_n||z|^{n-1}[|z+k-1| - k]$.

>0 if $|z+k-1| > k$

This shows that if $|z+k-1| > k$ then $Q(z) > 0$.

Hence all the zeros of Q(z) with $|z|>1$ lie in $|z+k-1| \le k$

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

3. Proof of Theorem 2

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.

Then the polar derivative of P(z) is given by $D_\alpha P(z) = n P(z) + (\alpha - z) P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$

$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.

Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that

$Q(z) = -[a_{n-1} + \alpha n a_n]z^n + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

$= -[a_{n-1} + \alpha n a_n][z+k-1]z^{n-1} + [k(a_{n-1} + \alpha n a_n) - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$

$+[(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$

$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$

$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$

$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$

$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-n} < 1$ for $i = 1, 2, 3, \ldots n-1$

Further

$|Q(z)| \ge |a_{n-1} + \alpha n a_n||z+k-1||z|^{n-1} - \{|k(a_{n-1} + \alpha n a_n) - 2a_{n-2} - \alpha(n-1)a_{n-1}||z|^{n-1}$

$+ \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}||z|^{m+1}$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m||z|^m$

$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}||z|^{m-1}$

$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|$
$+|na_0 + \alpha a_1|\}$.

$\geq |a_{n-1} + \alpha na_n|\,|z|^{n-1}[|z + k - 1| - |a_{n-1} + \alpha na_n|^{-1}\{|k(a_{n-1} + \alpha na_n) - 2a_{n-2}$
$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}|\,|z|^{-1} + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\,|z|^{-(n-m-2)}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m||\,z|^{-(n-m-1)}$
$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\,|z|^{-(n-m)} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|^{-(n-2)} + |na_0 + \alpha a_1|\,|z|^{-(n-1)}\}]$.

$\geq |a_{n-1} + \alpha na_n|\,|z|^{n-1}[|z + k - 1| - |a_{n-1} + \alpha na_n|^{-1}\{|k(a_{n-1} + \alpha na_n) - 2a_{n-2}$
$-\alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha ma_m|$
$+|(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| + |na_0 + \alpha a_1|\}]$.

$\geq |a_{n-1} + \alpha na_n|\,|z|^{n-1}[|z + k - 1| - |a_{n-1} + \alpha na_n|^{-1}\{-k(a_{n-1} + \alpha na_n) + 2a_{n-2}$
$+\alpha(n-1)a_{n-1} - 2a_{n-2} - \alpha(n-1)a_{n-1} + 3a_{n-3} + \alpha(n-2)a_{n-2} + \ldots$
$-(n-m-1)a_{m+1} - \alpha(m+2)a_{m+2} + (n-m)a_m + \alpha(m+1)a_{m+1}$
$-(n-m)a_m - \alpha(m+1)a_{m+1} + (n-m+1)a_{m-1} + \alpha ma_m$
$-(n-m+1)a_{m-1} - \alpha ma_m + (n-m+2)a_{m-2} + \alpha(m-1)a_{m-1} + \ldots$
$-(n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2 - (n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1$
$+|na_0 + \alpha a_1|\}]$.

$\geq |a_{n-1} + \alpha na_n|\,|z|^{n-1}[|z + k - 1| - |a_{n-1} + \alpha na_n|^{-1}\{-k(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}]$.
>0 if $|z + k - 1| > |a_{n-1} + \alpha na_n|^{-1}\{-k(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$

This shows that if
$|z + k - 1| > |a_{n-1} + \alpha na_n|^{-1}\{-k(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$ then $Q(z) > 0$.
Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z + k - 1| \leq |a_{n-1} + \alpha na_n|^{-1}\{-k(a_{n-1} + \alpha na_n) + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.
But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality and this completes the proof.

4. Proof of Theorem 3

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.Then the polar derivative of $P(z)$ is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$. Then

$D_\alpha P(z) = [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$
$+ [(n-m+1)a_{m-1} + \alpha ma_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots + [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.

As $\alpha = -a_{n-1}/na_n = -2a_{n-2}/(n-1)a_{n-1} = \ldots.. = -(n-m-1)a_{m+1}/(m+2)a_{m+2}$

$D_\alpha P(z) = [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$
$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha na_n]z^{n-1}$.

Now consider the polynomial $Q(z) = (1-z)\,D_\alpha P(z)$ so that

$Q(z) = -[(n-m)a_m + \alpha(m+1)a_{m+1}]z^{m+1} + [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1}$
$- \alpha ma_m]z^m + [(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1]$.

$= -[(n-m)a_m + \alpha(m+1)a_{m+1}]\,[z+k-1]\,z^m + [k\,\{(n-m)a_m + \alpha(m+1)a_{m+1}\}$
$- (n-m+1)a_{m-1} - \alpha ma_m]z^m + [(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2}$
$-\alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]\,z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1]$.

Now if $|z|>1$ then $|z|^{i-m} < 1$ for $i = n-1, n-2, \ldots.n-m$

Further,

$|Q(z)| \geq |(n-m)a_m + \alpha(m+1)a_{m+1}|\,|z+k-1|\,|z|^m - \{|k\,[(n-m)a_m + \alpha(m+1)a_{m+1}]$
$- (n-m+1)a_{m-1} - \alpha ma_m|\,|z|^m + |(n-m+1)a_{m-1} + \alpha ma_m - (n-m+2)a_{m-2}$
$-\alpha(m-1)a_{m-1}|\,|z|^{m-1} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|$
$+|na_0 + \alpha a_1|\}$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m\,[|z+k-1|$
$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{|k\,[(n-m)a_m + \alpha(m+1)a_{m+1}]$
$- (n-m+1)a_{m-1} - \alpha ma_m| + \ldots + |(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^{-(m-2)}$
$+|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1||z|^{-(m-1)} + |na_0 + \alpha a_1||z|^{-m}\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m\,[|z+k-1|$
$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{|k\,[(n-m)a_m + \alpha(m+1)a_{m+1}]$
$- (n-m+1)a_{m-1} - \alpha ma_m| + \ldots + |(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|$
$+|(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| + |na_0 + \alpha a_1|\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m\,[|z+k-1|$
$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{-k\,[(n-m)a_m + \alpha(m+1)a_{m+1}]$
$+(n-m+1)a_{m-1} + \alpha ma_m - \ldots - (n-2)a_2 - 3\alpha a_3 + (n-1)a_1 + 2\alpha a_2$
$-(n-1)a_1 - 2\alpha a_2 + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}]$.

$\geq |(n-m)a_m + \alpha(m+1)a_{m+1}||z|^m\,[|z+k-1|$
$-|(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{-k\,[(n-m)a_m + \alpha(m+1)a_{m+1}] + na_0 + \alpha a_1$
$+|na_0 + \alpha a_1|\}]$.
>0 if $|z + k - 1| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{-k\,(n-m)a_m - k\,\alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.

This shows that if
$|z + k - 1| > |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{-k\,(n-m)a_m - k\,\alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.
then $Q(z) > 0$.
Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z + k - 1| \leq |(n-m)a_m + \alpha(m+1)a_{m+1}|^{-1}\,\{-k\,(n-m)a_m - k\,\alpha(m+1)a_{m+1} + na_0 + \alpha a_1 + |na_0 + \alpha a_1|\}$.

But those zeros of Q(z) whose modulus is less than or equal to 1, already satisfy the above inequality since all the zeros of $D_\alpha P(z)$ are also the zeros of Q(z) as they lie in the circle defined by the above inequality and this completes the proof.

References

[1] G.Eneström, Remarqueesur un théorémrelatif aux racines de l'equation $a_n+....+a_0=0$ oütous les coefficient sontet positifs, TôhokuMath.J**18**(1920),34-36.
[2] S.KAKEYA, on the limits of the roots of an algebraic equation with positive coefficient, TôhokuMath.J**2**(1912-1913),140-142.
[3] A.Joyal, G.Labelle and Q.I.Rahman, On the location of zeros of polynomial, Canad. Math. Bull. **10** (1967),53-63.
[4] A.Aziz and B. A. Zargar, Some extensions of Eneström-KakeyaTheorem, GlasnikMatematiki, **31**(1996), 239-244..

Author Profile

G.L. Redd is from School of Mathematics and Statistics, University of Hyderabad - 500046, India

K. Anoosha is from School of Mathematics and Statistics, University of Hyderabad - 500046, India

K. Sravani is from School of Mathematics and Statistics, University of Hyderabad - 500046, India