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Sequence of Real Numbers

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Abstract: ***A sequence is an ordered list of things .In every days usage ,the word 'sequence' is used to suggest a succession of things or event arranged in some order such as chronological order, size order and so on. We are prepared to pursue question are more analytic nature, and begin with the study of convergence of sequence.***

Keywords: ϵ,≤,≥ infinity, General mathematical terms

1. Introduction

A sequence can be thought of as a list of element with a particular order. Sequences are useful in a number of mathematical disciplines for studying functions spaces and other mathematical structures using the convergence properties of sequence. These include the meaning of convergence sequence, type of sequence Cauchy criterion for convergence of sequence and use of sequences in different fields.

2. Sequence of Real Number

A sequence of real number is the collection of some map $f:N \to R$, when they are arranged in natural order of natural number.

f (1), f(2), f(3), ……….. f(n)…………

then we adopt a notation $< a_n >$= f(n)

such that

$< a_n >$={a_1,a_2 ,a_3 ,………….a_n……..}

= number of terms in a sequence is countable.

Range of sequence

The range of sequence $< a_n >$ is the set of values {a_n: n∈N} counting of distinct terms without repetition.

3. Type of Sequence

Bounded Sequence

A Sequence $< a_n >$ is said to be **bounded above** if there exist a real number K such that

a_n≤ K, ∀n∈N.

For example- a_n = $-n^2$<0 ∀n∈N. ⇒$< a_n >$ is bounded above.

A Sequence $< a_n >$ is said to be **bounded below** if there exist a real number k such that

k ≤ a_n , ∀n ∈N.

For example- a_n = nth prime

Since every prime number is ≥ 2,∴ a_n ≥ 2 ∀n

⇒$< a_n >$ is bounded below.

A Sequence $< a_n >$ is said to be **bounded** if there exist a real number k and K such that

k ≤ a_n≤ K , ∀n∈N.

For example- a_n = $\frac{1}{2^n}$ ∀n∈N.

Clearly 0<a_n<1/2 ∀n∈N

⇒$< a_n >$ is bounded

Unbounded Sequence

A Sequence $< a_n >$ is said to be un**bounded if** it is not bounded above and bounded below.

A sequence is unbounded if range is not bounded.

Oscillatory Sequence

A **seq**uence which is neither convergent nor-divergent is called oscillatory sequence.

Finite Oscillatory Sequence

A bounded sequence $< a_n >$ which is not convergent is said to oscillate finitely.

For example- a_n = $(-1)^n$ oscillate finitely since it is bounded and converges.

Infinite Oscillatory Sequence

An unbounded sequence $< a_n >$ which is diverges neither to +∞ and to - ∞ is said to oscillate infinitely.

For example- a_n =< $(-1)^n n^2$ > sequence oscillate infinitely.

Constant Sequence

A Sequence $< a_n >$ defined by a_n= c ∀n∈N is called a constant sequence.

Thus $< a_n >$={c, c, c……….c…} is a c is constant sequence with range {c} singleton.

For example- a_n=3

Monotonic Sequence

A Sequence $< a_n >$ is said to be **monotonic increasing** if $a_{n+1} \geq a_n$ ∀n∈N.

For example: a_n=n, $-n^2$ ∀n∈N

A Sequence $< a_n >$ is said to be **monotonic decreasing** if $a_{n+1} \leq a_n$ ∀n∈N.

For example - a_n=$\frac{1}{.2^n}$ ∀n∈N

Constant sequence is monotonic sequence

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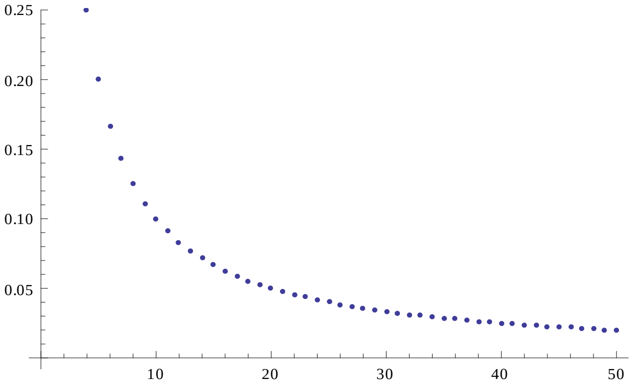
3.1 Limit Point of a sequence

One is the most important properties of a sequence is convergence. Informally, a sequence converges if it has a limit. Continuing informally, a (single infinite) sequence has a limit if it approaches some value L, called the limit, as n becomes very large.



Let $< a_n >$ be a sequence and l be a real number, we say l is the limit point of a sequence $< a_n >$ if for any ε > 0, (l-ε, l+ε) contains infinite members of $< a_n >$.

Or

$a_n \in (l-\varepsilon, l+\varepsilon)$ for infinite value of n.

For example - $a_n = <(-1)^n> = \{1,-1, 1,-1,\}$

(l-ε, l+ε) and (-l-ε, -l+ε), Limit point=1,-1

Because 1 and -1 repeats infinitely.

Convergent of Sequence

Let l be a real number and $< a_n >$ be a sequence .We say $< a_n >$ converge to l if for any ∊ > 0 there exist m∊ N

Such that $|a_n - l| < \epsilon$ ∀n≥ m

i.e. The distance between a_n and l can be made as small as we please.

i.e. For any ∊>0,

(l-ε, l+ε) leaves out any finite member of $< a_n >$

And we write $\lim a_n = l$.

If a sequence converges to same limit, then it is convergent; otherwise it is **divergent**

If a_n get arbitrary large number as $n \to \infty$ we write

$\lim_{n\to\infty} a_n = \infty$

In this case sequence a_n **diverges**, or converges to infinity.

If a_n becomes arbitrarily "small" negative number (large in magnitude) as $n \to \infty$ we write

$\lim_{n\to\infty} a_n = -\infty$

In this case sequence a_n **diverges**, or converges to minus infinity.

3.2 Properties of Convergent of Sequence

1. Every convergent sequence has a unique limit.

2. A sequence can not converge more than one limit point.

3. Every convergent sequence is bounded but not conversely.

For example : $a_n = <(-1)^n> = \{1,-1, 1,-1,\ldots\ldots.\}$

The sequence is bounded but not convergent.

4. If sequence converges $< a_n >$ is convergent the it converges to unique limit.

5. Every monotonic and bounded sequence is convergent.

(a) Every increasing sequence is convergent if it is bounded above.

(b) Every decreasing sequence is convergent if it is bounded below.

6. Every monotonic sequence either converges or diverges.

Or A monotonic sequence is never oscillatory.

7. (**Bolzano-Weierstrass**) Every bounded sequence has a limit point.

8. (Nested interval) Let $\{I_n = \{a_n, b_n\}\}$ be a sequence of closed interval,

(a) $I_{n+1} \subset I_n$ ∀n ∊N

(b) Length of nth interval = $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$

Then there exists a unique number x such that $x \in I_n$ ∀n.

i.e. $\bigcap_{n=1}^{\infty} I_n$ is singleton.

3.3 Application and important results

Important results for convergence and limits of (one-sided) sequence of real numbers include the following .These qualities are all true at least when both sides exist.

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

1. $\lim_{n\to\infty} k a_n = ka$ Where k is any constant.

2. $\lim_{n\to\infty} |a_n| = |a|$

3. $\lim_{n\to\infty} (a_n + b_n) = (a + b)$

Converse of the above is not true, the existence of $\lim_{n\to\infty} (a_n + b_n)$ does not implies that the two limit $\lim_{n\to\infty} a_n =$ a and $\lim_{n\to\infty} b_n =$b also exists.

For example : a_n=n, b_n=-n both sequence are divergent but $(a_n + b_n)$=n+-n=0 ∀n

so that $\lim_{n\to\infty} (a_n + b_n)$=0 exist.

4. $\lim_{n\to\infty} (a_n - b_n) = (a - b)$

Converse of the above is not true, the existence of $\lim_{n\to\infty} (a_n - b_n)$ does not implies that the two limit $\lim_{n\to\infty} a_n =$ a and $\lim_{n\to\infty} b_n =$b also exists.

For example - a_n= n, b_n= n both sequence are divergent but $(a_n - b_n)$=n - n=0 ∀ n

so that $\lim_{n\to\infty} (a_n - b_n)$ =0 exist.

5. $\lim_{n\to\infty} (a_n b_n) = ab$

Converse of the above is not true,

Let $a_n = b_n = (-1)^n$ both limit does not exist but

$a_n b_n = (-1)^n (-1)^n = (-1)^{2n}$ =1 ∀n

so that $\lim_{n\to\infty} (a_n b_n)$=**1 exist.**

6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided b ≠0.

7. If $a_n \le b_n$ ∀n ∊N, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

(Squeeze Theorem)If $a_n \le c_n \le b_n$ ∀n > N, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$=L, then $\lim_{n\to\infty} c_n$=L.

Properties on limit of a sequence

1. If $< a_n >$ converges to l ,then the sequence $\{x_n\}$ where

$x_n = \frac{a_1+a_2+\cdots\ldots\ldots+a_n}{n}$ Also converges to l.

Or

$$lim_{n\to\infty} a_n = l \Rightarrow lim_{n\to\infty} \frac{a_1+a_2+\cdots\ldots\ldots+a_n}{n} = l$$

2. If $< a_n >$ is a sequence of positive terms, then

$$Lt_{n\to\infty}(a_n)^{1/n} = Lt_{n\to\infty}\frac{a_{n+1}}{a_n}$$

Provided the limit on the RHS exist, finite or infinite.

4. Cauchy Sequences

A Cauchy sequence is a sequence whose term become arbitrarily close together as n gets very large. The notion of a Cauchy sequence is important in the study of sequence in the metric spaces; in particular, in real analysis is Cauchy characterization of convergence for sequence.

Definition-

A sequence $< a_n >$ is said to be Cauchy if for any $\in>0$ there exist $p\in N$ such that

$\left|a_{n+p} - a_n\right| <\in$ for all $n\in N$

A sequence is convergent iff it is Cauchy sequence.

Cauchy criterion for convergence gives an important method to claim that sequence is not convergent.

i.e. $|a_{2n} - a_n| > c$

$\Rightarrow < a_n >$ is not convergent.

Specifying a sequence by recursion

Sequences whose element is related to the previous elements in a straightforward way are often specified using **recursion** .This is in contrast to the specification of sequence elements in terms of their position.

The **Fibonacci sequence** can be defined using a recursive rule along with two initial elements. The rule is sequence that each element is the sum of the previous two elements, and the first two elements are 0 and 1.

$a_n = a_{n-1} + a_{n-2}$ with a_0=0 and a_1=1.

5. Use of Sequence in Different Field

Topology

Sequence plays an important role in topology especially in the study of metric space.

1) A metric space is compact exactly when it is sequentially compact.
2) A topological space is separable exactly when there is a dense sequence of points.

Sequence can be generalized to nets or filters.

Analysis

In analysis when we talking about sequence, one will generally consider sequence of the form

($a_1, a_2, a_3, \ldots. a_n \ldots\ldots$) or ($a_0, a_1, a_2, \ldots. a_n \ldots$)

Which is to say, infinitely sequences of elements indexed by natural number.

A sequence space is a vector space whose elements are infinite sequence of real or complex numbers. Equivalently, it is a function space whose element is functions from the natural number to the field K of real and complex. All sequence are linear subspace of this space .sequence space are typically equipped with a norm, or at least the structure of a topological vector space.

Abstract Algebra

Abstract algebra employs several type of sequence, including sequence of mathematical objects such as group or rings.

Linear Algebra

Sequences over a field may also be viewed as vectors in a vector space .Specifically; the set of F valued sequence. (Where F is field) is a function space (Intact, a product space) of F valued function over the set of natural number.

Set Theory

An ordinal indexed sequences is a generalization of sequences .If α is a limit ordinal and X is a set; an α-indexed sequences of elements of X is a function from α to X. In this terminology a w-indexed sequence is an ordinary sequence

6. Conclusion

Sequence are so important that algebra and analysis are incomplete without their mention. They provide a strong tool for researcher who are doing research in algebra particularly in linear algebra and in analysis specially in topology**.** Sequence are also of interest be study of patterns or puzzles ,such as in the study of prime numbers.

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