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Representation of Projection Operators on Fractal Measures in Weighted Modulation Spaces

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Abstract: ***Using a general wavelet system, we obtain upper bound estimate for a projection operator associated with fractal measures in a suitable weighted modulation space over R^n. Also a corollary, we derive the corresponding estimate in the Banach space of Feichtinger distributions.***

Keywords: Fractal measures,Wavelet Representation, Modulation spaces, Feichtinger algebra and Banach space of Feichtinger distributions..

1. Introduction

Strichartz has studied in detail (cf. [St 90a], [St 90b] and [St 91]) the nature of asymptotic behavior of fractal measures or distributions by means of their fractal expansion. In another paper [St 94] he has obtained a number of important results for fractal spectral asymptotics using wavelet expansions in the n-dimensional Euclidian space R^n. The general wavelet system used by Strichartz in this paper [St 94] is not necessarily orthogonal. He, in fact, has defined wavelet representations of a projection operator at level j and obtained upper bounds for it in L^p-spaces, $1 \le p \le \infty$, when $j > 0$ and $j < 0$. Also, he has determined pointwise bounds for projection operators and their partial sums.

Also, for a fractal measure μ, which is locally uniformly 0-dimensional, Strichartz (loc.cit), using Haar functions as orthonormal wavelet system, has obtained exact estimates in the $L^p(d\mu)$-space for the associated projection operators and their partial sum operators.

In the present paper our aim is to obtained upper bound estimates for the projection operators associated with fractal measures as defined by Strichartz in [St 94] in suitable weighted modulation spaces defined over R^n. It is well known that modulation spaces on locally compact abelian groups were, originally , defined by Feichtinger [Fei 83] as direct generalization of L^p-spaces. In section 2. we define moderate weight function associated with a submultiplicative weight function on R^n and use it to define the weighted Lebesgue space $L^p_w(R^n)$, $1 \le p \le \infty$, with respect to Lebesgue measure as usual.

In section 3, we define Short-Time Fourier Transform (STFT) of a function $f \in L^2(R^n)$ with respect to a non-zero window function $g \in L^2(R^n)$. Using STFT $V_g f$, we define modulations space $M^p_w(R^n)$ and its dual space $M^{p'}_{w'}(R^n)$, where $1/p + 1/p' = 1$ and $w' = w^{-1}$.

In the last section, following Strichartz (loc.cit.), we define a wavelet expansion of a function f with respect to the measure μ in the modulation space $M^1_w(R^n)$ and the wavelet system

$$\psi^{\lambda}_{j,k}(x) = 2^{nj/2}\ \psi^{\lambda}\ (2^j x - k),$$

where $\lambda \in \Lambda$ (a finite set), $j \in Z$, $k \in Z^n$ and Z^n is a set of n-tuples of integers.

In Theorem 4.1, we obtain asymptotic estimate for the projection operator $P_j(fd\mu)$ in the modulation space $M^{\infty}_{w'}(R^n)$, which is the dual space of $M^1_w(R^n)$. In case $w = 1$, we derive the corresponding results as a corollary in the space $S_0^{'}(R^n)$ of Feichtinger distributions.

2. Preliminaries

Let R^n be the n-dimensional Euclidian space and $\hat{R}^n$ its dual space. We assume that dx and $d\xi$ are Lebegue measures on R^n and $(\hat{R})^n$ respectively.

We define the translation and modulation operators T_x and M_ξ by their actions on a function f such that

$$T_x f(y) = f(y - x); \quad \forall\ x, y \in R^n,$$

and

$$M_\xi f(y) = f(t) \cdot e^{2\pi i y \xi}; \quad \forall\ y \in R^n \ and\ \xi \in \hat{R}^n.$$

A combination of the form $M_\xi T_x$ is known as time-frequency shift and, on account of the above definitions, we have

$$T_x M_\xi = e^{-2\pi i x \xi} M_\xi T_x$$

$\Rightarrow T_x$ and M_ξ commute if and only if $x \cdot \xi \in Z$, where Z is the set of all integers.

A weight function u on R^n is called submultiplicative if the following conditions hold :

(i) $u(0) = 1.$

and

(ii) $u(x+y) \le u(x)\, u(y)$ for all $x, y \in R^d$.

A weight functions w on R^d is known as u moderate provided:

$w(x+y) \le C\, u(x)\, w(y),$

C being a positive constant.

We suppose that $L_w^p(R^n), 1 \le p < \infty,$ is the Banach space of functions under the norm:

$$\| f \|_{p,w} = \| f \mid L_w^p \| = \left(\int_{R^n} | f(x)|^p \; w^p(x)\, dx \right)^{1/p}. \qquad (2.1)$$

The conjugate space of $L_w^p(R^n)$ is the space $L_{w'}^{p'}(R^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $w' = w^{-1}$

In case $p = \infty$, we define the Banach space $L_w^\infty(R^n)$ as the space of all measurable functions f such that

$$\| f \|_{\infty,w} = ess\sup\{| f(x) | w(x) : x \in R^n\} < \infty.$$

It is well known that $L_w^p(R^n)$ is translation invariant and $L_w^1(R^n)$ is a commutative Banach algebra with respect to convolution. Also, we have

$L_w^p * L_w^1 \subseteq L_w^p$

and

$\| f * g \|_{p,w} \le \| f \|_{p,w} \; \| g \|_{1,w}$.

3. Modulation Space on R^n.

Let $g \ne 0$ be a fixed window function on R^n and $g \in L^2(R)$. Then the short-time Fourier transform (STFT) of a function $f \in L^2(R^n)$ with respect to g is defined by

$$V_g f(x,\xi) = \int_{R^n} f(y) \overline{g\,(y-x)} e^{-2\pi i y \xi} dy \qquad (3.1)$$

for all $x \in R^n$ and $\xi \in \hat{R}^n$.

We suppose that $S(R^n)$ is the Schwartz class consisting of all test functions or $C^\infty -$functions f on R^n with compact support such that

$$\sup_{x \in R^d} | D^\alpha \; X^\beta \; f(x) | < \infty \quad forall \;\; \alpha, \beta \in Z_+^n,$$

where

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}, \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \ldots\ldots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \qquad |\alpha| = \sum_{j=1}^{d} \alpha_j.$$

$X^\alpha f(x) = x^\alpha f(x)$ is the multiplicative operator and Z_+^n is the set of all positive integral n-tuples.

Let $S'(R^n)$ be the dual space of $S(R^n)$. Element is $S'(R^n)$ are known as tempered distributions. In this section, on the line of Feichtinger [Fei 83], we define weighted modulation space $M_w^p(R^n)$. We assume that g is a non-zero fixed window function of the class $S(R^n)$, w is a $u-$moderate weight function on R^n. Then the weighted modulation space $M_w^p(R^n)$, $1 \le p \le \infty$, is define by

$$M_w^p(R^n) = \{ f \in S'(R^n) : V_g f \in L_w^p(R^n \times \hat{R}^n)\} \qquad (3.2)$$

It is known that $M_w^p(R^n)$ is a Banach space with respect to the norm

$$\| f \mid M_w^p \| = \| V_g f \mid L_w^p(R^n \times \hat{R}^n) \|. \qquad (3.3)$$

The dual space of $M_w^p(R^n)$ is the space $M_{w'}^{p'}(R^n)$. In case $p = 1$ *and* $w = 1$, the space $M_w^p(R^n)$ reduces to the well known Feichtinger algebra $S_0(R^n)$, which is a Banach space satisfying some very important functorial properties. The dual space of $S_0(R^n)$ is denoted by $S_0'(R^n)$, which is known as Banach space of Feichtinger distribution (for detail see [Fei 81], page. 281)and $S_0(R^n)$, contains $S'(R^n)$ as dense subspace. Also, we have the continuous embeddings:

$$S_0(R^n) \hookrightarrow S'_0(R^n)$$

4. Wavelet Representation of Projection Operators

Following Strichartz [St 94, p.270], we say that a measure μ on R^n is locally uniformly α -dimensional provided the estimate

$$\mu(B_r(x)) \le C \; r^\alpha$$

for all $x \in R^n$ and all $r \le 1$, where $B_r(x)$ is the ball of radius r with center x. Any such measure may be expressed in the form

$$d\mu = f \; d\mu_\nu + \nu,$$

where μ_α denotes α -dimensional Hausdorff measure and $\nu << \mu_\alpha$, which means that $\mu_\alpha(E) < \infty \Rightarrow \nu(E) = 0.$

Let μ be any locally uniformly α-dimensional measure over R^n and let $f \in M_w^p(R^n)$. Then the wavelet representation coefficients associated with $f \in M_{w^{-1}}^{\infty}$ is given by

$$\langle f d\mu, \psi_{j,k}^{\lambda} \rangle = \int_{R^n} f(x) \psi_{j,k}^{\lambda}(x)\, d\mu(x), \tag{4.1}$$

where $M_{w^{-1}}^{\infty}(R^n)$ is dual space of $M_w^1(R^n)$,

$$\psi_{j,k}^{\lambda}(x) = 2^{nj/2} \quad \psi^{\lambda}(2^j x - k). \tag{4.2}$$

is the wavelet system generated by the functions ψ^{λ} indexed by $\lambda \in \Lambda$ (a finite set), $j \in Z$, $k \in Z^n$ and Z^n is the set of all n-tuples of integers.

In this section we assume that $\psi^{\lambda}(x)$, $\lambda \in \Lambda$, are the real valued measurable functions on R^d such that

$$|\psi^{\lambda}(x)| \le C\,(1 + |x|)^{-n-\varepsilon}, \quad \varepsilon > 0, \tag{4.3}$$

where C is a positive constant not necessarily the same at each occurrence. It may be mentioned here that the wavelet system $\{\psi_{j,k}^{\lambda}(x)\}$ is not necessarily orthogonal or complete.

Also, the integral on the right-hand side of (4.1) is well defined provided

$$\psi_{j,k}^{\lambda} \in M_w^1(R^n).$$

In order to verify (4.3), we consider $\bigcup_{m \in Z^n} Q_{0,m}$ as the standard cube tiling of R^n.

Since

$$\mu(Q_{0,m}) \le C \quad \forall\ m,$$

we have

$$\|\psi_{j,k}^{\lambda} \mid M_{w'}^{p'}\| = \sum_m \int_{Q_0,m} \|\psi_{j,k}^{\lambda} \mid M_{w'}^{p'}\|\, d\mu$$

$$\le C \sum_m \left(\sup_{Q_0,m} \|\psi_{j,k}^{\lambda} \mid M_{w'}^{p'}\| \right) \quad by \quad (4.2)$$

$$< \infty.$$

Now, on the lines of Strichartz [St 94, p. 272], we write the formal representation of the projection operators P_j associate with f at level j in the form

$$P_j\ (f\ d\mu)(x) = \sum_{\lambda,k} \langle f\ d\mu, \psi_{j,k}^{\lambda} \rangle\ \ \psi_{j,k}^{\lambda}(x). \tag{4.4}$$

In this section we prove the following theorem which corresponds to a results of Strichartz [ST 94, Theorem 2.1]:

Theorem 4.1. If μ is a locally uniformly α-dimensional measure and $f \in M_{w^{-1}}^{\infty}(R^n)$, then $P_j(f\ d\mu)$ is a well defined function in $M_{w^{-1}}^{\infty}(R^n)$ and

$$\|P_j(f\ d\mu) \mid M_{w^{-1}}^{\infty}\| \le C \|f \mid M_{w^{-1}}^{\infty}(d\mu)\| \quad for \quad j \ge 0.$$

Proof: For $j \ge 0$, we have

$$\|P_j(f\ d\mu)(x) \mid M_{w^{-1}}^{\infty}\| = \|\sum_{\lambda,k} \int_{R^n} \psi_{j,k}^{\lambda}(x) \overline{\psi_{j,k}^{\lambda}(y)}\ f\ d\mu\|_{M_{w^{-1}}^{\infty}}$$

$$\le 2^{nj} \|\sum_{j,k} \int_{R^n} |\psi^{\lambda}(2^j x - k)| \cdot |\psi^{\lambda}(2^j y - k)| \cdot |f|\, d\mu\|_{M_{w^{-1}}^{\infty}}$$

$$\le 2^{nj} \|f \mid M_{w^{-1}}^{\infty}(d\mu)\| \sum_{j,k} \int_{R^n} \int_{R^n} |\psi^{\lambda}(2^j x - k)|$$

$$\cdot |\psi^{\lambda}(2^j y - k)|\ d\mu(y)\, dx.$$

But, using (4.3), we see that

$$2^{nj} \int_{R^n} \psi^{\lambda}(2^j x - k)|\, dx = \|\psi^{\lambda}\|_1 < \infty \quad by \quad (4.3).$$

Hence, we see that

$$\|P_j(f\ d\mu)(x) \mid M_{w^{-1}}^{\infty}\| \le C \|f \mid M_{w^{-1}}^{\infty}(d\mu)\|\ \|\sum_{\lambda,k} |\psi^{\lambda}(2^j y - k)|\|_{\infty}$$

$$\le C \|f \mid M_{w^{-1}}^{\infty}(d\mu)\|,$$

because

$$\|\sum_{\lambda,k} |\psi^{\lambda}(2^j y - k)|\|_{\infty}$$

is independent of j and finite by (4.3).

This complete the proof of the theorem.

Corollary. If μ is a locally uniformly $\alpha-$dimensional measure and $f \in S_0'(R^n)$, then $P_j(f\ d\mu)$ is a well defined function in $S_0'(R^n)$ and

$$\|P_j\ (f\ d\mu) \mid S_0'\| \le c \|f \mid S_0'\ (d\mu)\| \quad for\ j \ge 0.$$

The proof follows from the above theorem for $w \equiv 1$.

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