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# Albertson's Conjecture on Crossing Numbers

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**Abstract:** ***In this paper, we prove the Albertson's Conjecture which states that "If graph $G$ has chromatic number $r$, then the crossing number of $G$ is atleast that of the $K_r$" for $7 \leq r \leq 10$ using results of Dirac, Gallai and Kostochka Stiebitz and Pach et al.***

**Keywords:** Crossing number, complete graph, critical graph, lower bound.

## 1. Introduction

The history of graph theory may be specifically traced to 1735, when the Swiss Mathematician Leonard Euler solved the "Konigsberg Bridge" problem. One of the famous problems in graph theory is the "Four Color Conjecture" proposed by Francis Guthrie, a student of Augustus DeMorgan, around 1850. The study and generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the coloring of the graphs embedded on surfaces with arbitrary genus. The four color problem has played a leading role in the development of graph theory for more than a century.

There are three classic relaxations of planarity. The first is that of a graph embedded on an arbitrary surface. The second classic relaxation of planarity is thickness, the minimum number of planar subgraphs needed to partition the edges of the graph. The third classic relaxation of planarity is crossing number. The first question about the connection between the chromatic number and the crossing number is whether the chromatic number is bounded by a function of the crossing number. Alberson's [2] conjectured that $\chi(G) = O(cr(G)^{\frac{1}{4}})$ and this was known by Schaefer.

Guy conjectured [7] the crossing number of the complete graph as $cr(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$
He verified the conjecture for $n \leq 10$ and Pan and Richter [17] confirmed it for $n = 11,12$. The right hand side of the equation is denoted as $f(n)$.

Kleitman proved that $\lim_{n\to\infty} \frac{cr(K_n)}{f(n)} \geq 0.80$ [9]. De Klerk et al [10] strengthened this lower bound to 0.83. By redefining the techniques in [10], de Klerk, Pasechnik and Schrijver [11] further improved the lower bound to 0.8594 .

An easy application of the four color theorem shows that if $cr(G) = 1$, then $\chi(G) \leq 5$. Oporowski and Zhao [15] showed that the conclusion also holds when $cr(G) = 2$. They further showed that if $cr(G) = 3$ and $G$ does not contain a copy of $K_6$ , then $\chi(G) \leq 5$; they conjectured that this conclusion remains true even if $cr(G) \in \{4,5\}$ . Albertson, Heenehan, McDonough, and Wise showed that if $cr(G) \leq 6$, then $\chi(G) \leq 6$.

The relation between pair of crossings was first studied by Albertson [2]. Wenger[18] and Harmon [8] showed that any graph with four independent crossings has an independent set of vertices one from each cluster, but their exists a graph with five independent crossings that contains no independent set of vertices one from each cluster. Finally, Kral and Stacho [13] proved the conjecture that if $G$ has a drawing in which all crossings are independent, then $\chi(G) \leq 5$. At an AMS special session in Chicago in October of 2007, Albertson conjectured the following :
If $\chi(G) \geq r$, then $cr(G) \geq cr(K_r)$

Dirac [4] verified the conjecture for $r \leq 4$. In the case $r = 5$, the conjecture is equivalent to the four color theorem. Oporowski and Zhao verified for $r = 6$. The statement of this problem is similar to that of the Heawood problem. $cr(K_n)$ is known only for $n \leq 12$ and the results for $n = 11, 12$ are quiet recent. This paper is organized as follows. In Section 2 we discuss known lower bounds on the number of edges in $r$ critical graphs. In Section 3 we discuss known lower bounds on the crossing number, in terms of the number of edges. In Section 4 we prove Albertson's conjecture for $7 \leq r \leq 10$ by combining the results in the previous sections.

## 2. Color Critical Graphs

The concept of color criticality in order to simplify graph coloring theory was introduced by Dirac. Let $G$ denote an $r$-critical graph with $n$ vertices and $m$ edges.

The excess $\epsilon_r(G)$of the graph $G$ is

$$\epsilon_r(G) = 2m - (r-1)n \qquad (1)$$

Since $G$ is $r$-critical, then every vertex has degree atleast $r-1$and so $\epsilon_r(G) \geq 0$.

Brook's theorem is equivalent to saying that equality holds if and only if $G$ is complete or an odd cycle.

### 2.1 Dirac's Bound [5]

Dirac proved that for $r \geq 3$ , if $G$ is not complete ,then $\epsilon_r(G) \geq r-3$.
From (1)

$$\epsilon_r(G) = 2m - (r-1)n$$
$$\Longrightarrow \; r-3 \leq 2m - (r-1)n$$
$$\Longrightarrow \; \frac{r-3}{2} + \frac{r-1}{2}n \leq m$$

Therefore if $G$ is $r$-critical and not a complete graph and $r \geq 3$, then

$$m \geq \frac{r-1}{2}n + \frac{r-3}{2} \qquad (2)$$

He later gave a complete characterization for $r \geq 4$ of those $r$-critical graphs with excess $r-3$, and, in particular they all have $2r-1$ vertices.

### 2.2 Gallai's Bound [6]

Gallai proved that $r$-critical graphs that are not complete and that have atmost $2r-2$ vertices have much larger excess. Namely if $G$ has $n = r + p$ vertices and
$2 \leq p \leq r-2$, then $\epsilon_r(G) \geq pr - p^2 - 2$.
From (1)

$$\epsilon_r(G) = 2m - (r-1)n$$
$$\Rightarrow pr - p^2 - 2 \leq 2m - (r-1)n$$
$$\Rightarrow \frac{pr - p^2 - 2}{2} + \frac{r-1}{2}n \leq m$$

Therefore if $G$ is $r$-critical and not a complete graph and $r \geq 3$, then

$$m \geq \frac{r-1}{2}n + \frac{pr - p^2 - 2}{2} \tag{3}$$

A fundamental difference between Gallai's bound and Dirac's bound is that Gallai's bound grows with the number of vertices while Dirac's does not.

### 2.3 Kostochka's and Stiebitz's Bound [12]

Kostochka and Stiebitz proved that if $n \geq r + 2$ and $n \neq 2r - 1$, then $\epsilon_r(G) \geq 2r - 6$.
From (1)

$$\epsilon_r(G) = 2m - (r-1)n$$
$$\Rightarrow 2r - 6 \leq 2m - (r-1)n$$
$$\Rightarrow (r-3) + \frac{r-1}{2}n \leq m$$

Therefore if $G$ is $r$-critical, $n \geq r + 2$ and $n \neq 2r - 1$, then

$$m \geq \frac{r-1}{2}n + (r-3) \tag{4}$$

We finish the section with a simple lemma classifying the $r$ critical graphs with at most $r + 2$ vertices.

**Lemma 1**
For $r \geq 3$, the only $r$-critical graphs with atmost $r+2$ vertices are $K_r$ and $K_{r+2} \setminus C_2$, the graph obtained from $K_{r+2}$ by deleting the edges of a cycle of length five.

## 3. Lower bounds on crossing number [1],[14],[16]

A simple sequence of Euler's Polyhedral formula is that, "Every planar graph with $n \geq 3$ vertices have atmost $3n-6$ edges".

Suppose $G$ is a graph with $n$ vertices and $m$ edges. By deleting one crossing edge at a time from a drawing of $G$ until no crossing edge exist, then

$$cr(G) \geq m - (3n - 6) \tag{5}$$

The inequality is best when $m \leq 4(n-2)$

**Pach *et al* lower bounds on the crossing number of graphs in terms of the number of edges and vertices.**

When $4(n-2) \leq m \leq 5(n-2)$

$$cr(G) \geq \frac{7}{3}m - \frac{25}{3}(n-2) \tag{6}$$

When $5(n-2) \leq m \leq 5.5(n-2)$

$$cr(G) \geq 3m - \frac{35}{3}(n-2) \tag{7}$$

When $m \geq 5.5(n-2)$

$$cr(G) \geq 4m - \frac{103}{6}(n-2) \tag{8}$$

The crossing lemma states that,
The crossing number of every graph $G$ with $n$ vertices and $m \geq 4n$ edges satisfies

$$cr(G) \geq \frac{1}{64}\frac{m^3}{n^2} \tag{9}$$

Using (8) for $m \geq \frac{103}{16}n$,

$$cr(G) \geq \frac{1}{31.1}\frac{m^3}{n^2} \tag{10}$$

## 4. Albertson's Conjecture for $7 \leq r \leq 10$

In this section we prove Albertson's conjecture for $r = 7,8,9,10$. Note that if H is a subgraph of $G$, then $cr(H) \leq cr(G)$. Therefore, to prove Albertson's conjecture for a given $r$, it suffices to prove it only for $r$-critical graphs.

**Proposition 1**
If $\chi(G) = 7$, then $cr(G) \geq 9 = cr(K_7)$

**Proof**
Suppose $G$ is 7-critical. Let $n$ be the number of vertices of $G$ and $m$ be the number of edgesof $G$.
By Dirac's bound,

$$m \geq \frac{r-1}{2}n + \frac{r-3}{2}$$
$$\geq \frac{7-1}{2}n + \frac{7-3}{2} \geq 3n + 2$$

If a graph has a drawing in the plane in which each edge intersects atmost one other edge, then the graph has chromatic number atmost 6.

Consider a drawing $D$ of $G$ in the plane with $cr(G)$ crossings. Since $G$ has chromatic number 7, there is an edge $e$ in $D$ that intersects atleast two other edges.

Beginning with $e$, delete one crossing edge at a time, until no crossing edges exist, then

$$cr(G) \geq m - (3n - 6) + 1$$
$$\geq 3n + 2 - 3n + 6 + 1$$
$$\geq 9$$

Hence if $\chi(G) = 7$, then $cr(G) \geq 9 = cr(K_7)$.

**Proposition 2**
If $\chi(G) = 8$ and $G$ does not contain $K_8$, then

$$cr(G) \geq 20 > 18 = cr(K_8)$$

**Proof**
Suppose $G$ is 8-critical.
Let $n$ be the number of vertices and $m$ be the number of edges of $G$.

$$n \geq r + 2 \geq 8 + 2 \geq 10$$

When$n = 2r - 1 = 16 - 1 = 15$,
the Dirac's bound gives

$$m \geq \frac{r-1}{2} n + \frac{r-3}{2}$$
$$\geq \frac{8-1}{2}(15) + \frac{8-3}{2}$$
$$\geq 55$$

Since $m \geq 55$, the inequality (6)gives

$$cr(G) \geq \frac{7}{3} m - \frac{25}{3}(n-2)$$
$$\geq \frac{7}{3}(55) - \frac{25}{3}(15-2)$$
$$\geq 20$$

When $n \neq 15$, the bound of Kostochka and Stiebitz gives

$$m \geq \frac{r-1}{2}(n) + r - 3$$
$$\geq \frac{8-1}{2}(n) + 8 - 3$$
$$\geq \frac{7}{2} n + 5$$

Thus the inequalities (5)&(6)gives

$$(5) \implies \quad cr(G) \geq m - (3n - 6)$$
$$\geq \frac{7}{2} n + 5 - 3n + 6$$
$$\geq \frac{n}{2} + 11$$
$$(6) \implies \quad cr(G) \geq \frac{7}{3} m - \frac{25}{3}(n-2)$$
$$\geq \frac{49}{6} n + \frac{35}{3} - \frac{25}{3} n + \frac{50}{3}$$
$$\geq -\frac{n}{6} + \frac{85}{3}$$

When $n \geq 18$, the first lower bound shows

$$cr(G) \geq \frac{n}{2} + 11$$
$$\geq \frac{18}{2} + 11$$
$$\geq 20$$

When $n \leq 50$, the second lower bound shows

$$cr(G) \geq -\frac{n}{6} + \frac{85}{3}$$
$$\geq -\frac{50}{6} + \frac{85}{3}$$
$$\geq 20$$

Hence if $\chi(G) = 8$, then $cr(G) \geq 20 > 18 = cr(K_8)$.

**Proposition 3**
If $\chi(G) = 9$ and $G$ does not contain $K_9$, then

$$cr(G) \geq 41 > 36 = cr(K_9)$$

**Proof**
Suppose $G$ is 9-critical.
Let $n$ be the number of vertices and $m$ be the number of edgesof $G$.

$$n \geq r + 2 \geq 9 + 2 \geq 11$$

When $n = 2r - 1 = 2(9) - 1 = 17$,
the Dirac's bound gives

$$m \geq \frac{r-1}{2}(n) + \frac{r-3}{2}$$
$$\geq \frac{9-1}{2}(17) + \frac{9-3}{2}$$
$$\geq 71$$

When$m \geq 71$,the inequality (6)gives

$$cr(G) \geq \frac{7}{3} m - \frac{25}{3}(n-2)$$
$$\geq \frac{7}{3}(71) - \frac{25}{3}(17-2)$$
$$\geq 41.3$$

When $n \neq 17$, the bound of Kostochka and Stiebitz gives

$$m \geq \frac{r-1}{2}(n) + r - 3$$
$$\geq \frac{9-1}{2}(n) + 9 - 3$$
$$\geq 4n + 6$$

When $m \geq 4n + 6$, the inequality(6)gives

$$cr(G) \geq \frac{7}{3} m - \frac{25}{3}(n-2)$$
$$\geq \frac{7}{3}(4n+6) - \frac{25}{3}(n-2)$$
$$\geq \frac{92}{3} + n$$

If $n \geq 11$, the lower bound shows that

$$cr(G) \geq \frac{92}{3} + 11 \geq 41$$

Hence if $\chi(G) = 9$, then $cr(G) \geq 41 > 36 = cr(K_9)$.

**Proposition 4**
If $\chi(G) = 10$ and $G$ does not contain $K_{10}$, then

$$cr(G) \geq 69 > 60 = cr(K_{10})$$

**Proof**

Suppose $G$ is 10-critical.
Let $n$ be the number of vertices and $m$ be the number of edges of $G$.

$$n \geq r + 2 \geq 10 + 2 \geq 12$$

When $n = 2r - 1 = 2(10) - 1 = 19$,

the Dirac's bound gives

$$m \geq \frac{r-1}{2}(n) + \frac{r-3}{2}$$
$$\geq \frac{10-1}{2}(19) + \frac{10-3}{2}$$
$$\geq 89$$

When $m \geq 109$, the inequality (7) gives

$$cr(G) \geq 3m - \frac{35}{3}(n-2)$$
$$\geq 3(89) - \frac{35}{3}(19-2)$$
$$\geq 69$$

When $n \neq 19$, the bound of Kostochka and Stiebitz gives

$$m \geq \frac{r-1}{2}(n) + r - 3$$
$$\geq \frac{10-1}{2} n + 10 - 3$$
$$\geq \frac{9}{2} n + 7$$

Then inequality (8) gives

$$cr(G) \geq 4m - \frac{103}{6}(n-2)$$

$$\geq 4\left(\frac{9}{2}n+7\right)-\frac{103}{6}(n-2)$$
$$\geq \frac{5}{6}n+\frac{187}{3}$$

If $n \geq 12$, the lower bound gives

$$cr(G) \geq \frac{5}{6}(12)+\frac{187}{3}$$
$$\geq 72$$

Hence if $\chi(G) = 10$, then $cr(G) \geq 69 > 60 = cr(K_{10})$.

## 5. Conclusion

In this paper we have proven the Albertson's conjecture for $7 \leq r \leq 10$ using the lower bounds on the number of edges of critical graphs and lower bounds on crossing number.

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