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K-Even Even Edge Graceful Labeling and Some Complementary Graceful Labeling

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Abstract: ***S.P Lo [4] introduced the notion of edge-graceful graphs. Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang[6] introduced the k-edge-graceful graphs. B. Gayathri, M. Duraisamy and M. Tamilselvi [3] introduced the even edge-graceful graphs. In this paper, we introduce definitions of k-even even edge gracefulness, complementary odd-even graceful labeling, complementary edge-odd graceful labeling and we also prove that some well known graphs namely, Friendship graph F_m, prism D_n, $C_m \times C_n$ etc., are k-even even edge graceful.***

Keywords: k-even even edge graceful labeling, complementary edge-odd graceful labeling and odd-even graceful labeling.

1. Introduction

Let G be a simple undirected graph with p vertices and q edges. Most graph labeling methods trace their origin to one introduced by Rosa [8] in 1967, or the one given by Graham and Sloane in 1980. S.P. Lo, introduced the notion of edge-graceful graphs. Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the k-edge-graceful graphs. B. Gayathri, M. Duraisamy and M.Tamilselvi[3] introduced the even edge-graceful graphs. We have introduced a labeling called k- even even edge graceful labeling and have also introduced complementary odd-even graceful labeling and complementary edge-odd graceful labeling.

Definition 1.1: A graph is k-even even edge graceful (k>0) if there exists an injective map f: E(G)→ {2k,2k+2,…2k+2q-2} so that the induced map f^*: V(G) →{0, 2,…,(2z-2)} defined by $f^*(x) \equiv \Sigma f(xy) \pmod{2z}$ where z = max {p, q} makes all distinct and even.

Definition 1.2: If f is an odd-even graceful labeling of a graph G = (V, E) with q edges, then the labeling φ defined by φ (v) = (2q+2) - f (v) for all v∈V(G) is again an odd-even graceful labeling of G and is called complementary odd-even graceful labeling.

Definition 1.3: If f is an edge-odd graceful labeling of a graph G = (V, E) with q edges, then the labeling φ defined by φ (e) = 2q- f(e) for all e ∈ E(G) is again an edge-odd graceful labeling of G and is called complementary edge-odd graceful labeling.

A necessary condition: If the (p,q) graph G is k-even even edge graceful, then $q(q+2k-1) \equiv 0 \pmod z$,where z = max {p, q}.
Remark: 1-even even edge graceful graph is an even even edge graceful.

2. Main Results

Definition 2.1: A friendship graph F_m $(m \geq 2)$ is the one point union of m cycles of length 3.

Theorem 2.1.1: The Friendship graph F_m is k-even even edge graceful if m is odd.
Proof: Let the vertex set be $V = \{v, v_i \mid 1 \leq i \leq 2m\}$ and the edge set be

$$E = \{e_i = vv_j \mid 1 \leq j \leq 2m \text{ and } i \equiv 1,2 \pmod 3\} \cup \{e_i = v_j v_{j+1} \mid j \text{ is odd and } i \equiv 0 \pmod 3\}$$

Clearly |V| = 2m+1and |E |= 3m.

Define f: E(G) → {2k, 2k+2, …, 2k+2q-2} as follows:
Case (1): $k \equiv 0 \pmod 3$
$f(e_1) = 2k+6m-4$, $f(e_2) = 2k+6m-2$, $f(e_3) = 2k$ and
$f(e_i) = 2k+(2i-6)$; i = 4,5,…,2m.
Case (2): $k \equiv 1 \pmod 3$
$f(e_1) = 2k$, $f(e_2) = 2k+2$, $f(e_3) = 2k+4$ and $f(e_i) = 2k+(2i-2)$; i = 4,5,…,2m
Case (3): $k \equiv 2 \pmod 3$
$f(e_1) = 2k+6m-2$, $f(e_2) = 2k$, $f(e_3) = 2k+2$ and $f(e_i) = 2k+(2i-4)$; i = 4,5,…,2m. Thus the induced vertex labels are:
Case (1): $k \equiv 0 \pmod 3$
$f(v_i) \equiv 4k+6i-10 \pmod{6m}$ i = 3,5,…,2m-1;
$f(v_i) \equiv 4k+6i-14 \pmod{6m}$ i = 2,4,…,2m & f(v) = 0.
Case (2): $k \equiv 1, 2 \pmod 3$
$f(v_i) \equiv 4k+6i-2 \pmod{6m}$ i = 3,5,…,2m-1;
$f(v_i) \equiv 4k+6i-6 \pmod{6m}$ i = 2,4,…,2m and f(v) = 0.

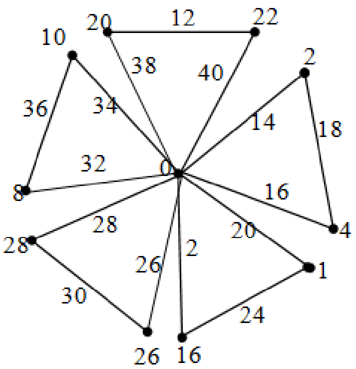


Figure 1: 6–even even edge gracefully labeled Friendship graph F_5.

Definition 2.2: For $n \geq 3$, prism D_n is the Cartesian product $C_n \times K_2$ where Cn is a cycle on n-vertices and k_2 is the complete graph on 2-vertices.

Theorem 2.2.1: Prism D_n is k-even even edge graceful.

Proof: Let G be a Prism graph with 2n vertices and 3n edges. Let $\{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ be the set of vertices and edges

$$e_i = \begin{cases} v_n v_1 & for\ i = 1 \\ v_{i-1} v_i & for\ 2 \le i \le n \quad and \quad n+2 \le i \le 2n \\ v_{2n} v_{n+1} & for\ i = n+1 \\ v_{n+1} v_1 & for\ i = 2n+1 \\ v_{2n-j} v_{n-j} & for\ 2n+2 \le i \le 3n \quad and \quad 0 \le j \le n-2 \end{cases}$$

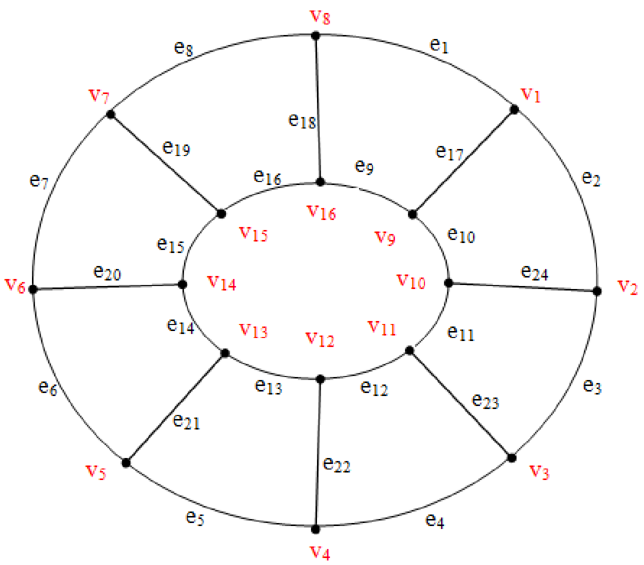


Figure 2

First, we label the edges as follows:
Define $f(e_i) = 2k+2i-2$ for $1 \le i \le 2n$.
Then the induced vertex labels are as follows:
Case (1): $f(v_1) = f(e_1)+f(e_2)+f(e_{2n+1})$
$\equiv 2k + 2 - 2 + 2k + 4 - 2 + 2k + 2(2n+1) - 2 \pmod{6n}$
$\equiv 6k + 2 + 4n \pmod{6n}$
Case (2): $f(v_i) = f(e_i)+f(e_{i+1})+f(e_{3n-i+2})$ for $2 \le i \le n-1$
$\equiv 2k+2i-2+2k+2(i+1)-2+2k+2(3n-i+2)-2 \pmod{6n}$
$\equiv 2k+2i-2+2k+2i+2-2+2k+6n-2i+4-2 \pmod{6n}$
$\equiv 6k+2i \pmod{6n}$ for $2 \le i \le n-1$
Induced vertex labels are {6k +4, 6k +6, …, 6k+2(n-1)}.
Case (3): $f(v_n) = f(e_n)+f(e_1)+f(e_{2n+2})$
$\equiv 2k+2n-2+2k+2-2+2k+2(2n+2)-2 \pmod{6n}$
$\equiv 2k+2n-2+2k+2k+4n+4-2 \pmod{6n}$
$\equiv 6k \pmod{6n}$
Case (4): $f(v_{n+1}) = f(e_{n+1})+f(e_{n+2})+f(e_{2n+1})$
$\equiv 2k+2n+2-2+2k+2n+4-2+2k+4n+2-2 \pmod{6n}$
$\equiv 2k+2n+2k+2n+2+2k+4n \pmod{6n}$
$\equiv 6k+2n+2 \pmod{6n}$
Case (5):
$f(v_i) = f(e_i)+f(e_{i+1})+f(e_{4n-i+2})$ for $n+2 \le i \le 2n-1$
$\equiv 2k+2i-2+2k+2(i+1)-2+2k+2(4n-i+2)-2 \pmod{6n}$
$\equiv 2k+2i-2+2k+2i+2-2+2k+8n-2i+4-2 \pmod{6n}$
$\equiv 6k+2i+2n \pmod{6n}$ for $n+2 \le i \le 2n-1$
induced vertex labels are {6k+4n +4, 6k+4n +6, …, 6k-2}.
Case (6): $f(v_{2n}) = f(e_{2n})+f(e_{n+1})+f(e_{2n+2})$
$\equiv 2k+4n-2+2k+2(n+1)-2+2k+2(2n+2)-2 \pmod{6n}$
$\equiv 2k+4n-2+2k+2n+2-2+2k+4n+4-2 \pmod{6n}$
$\equiv 6k+4n \pmod{6n}$

Hence induced vertex labels of the graph are
[{6k+2+4n} ∪ {6k+4, 6k +6, …, 6k+2(n-1)} ∪ {6k} ∪ {6k+2n+2 } ∪ {6k+4n +4, 6k+4n +6, …, 6k-2} ∪ {6k+4n}](mod 6n).
Hence Prism D_n is k -even even edge graceful.

Illustration: Figure. 3 shows 14 -even even edge graceful labeling of Prism D_8.

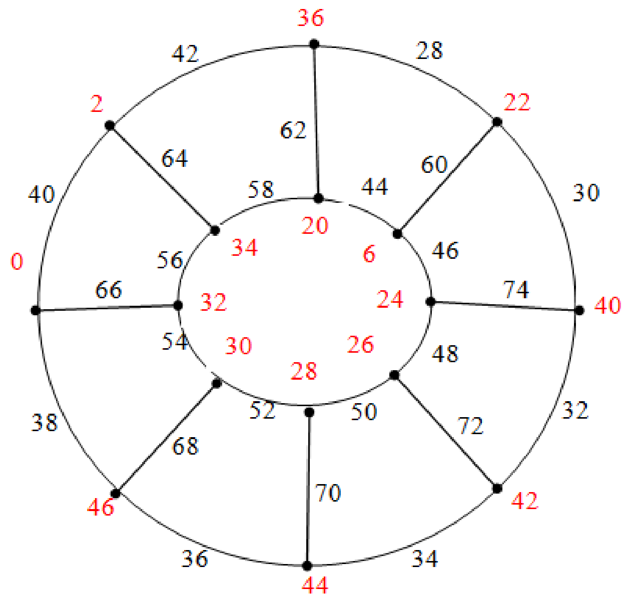


Figure 3

Theorem 2.3: The graph $C_m \times C_n$ is k-even even edge graceful.

Proof: Let the web graph $C_m \times C_n$ be a graph with mn vertices and $2mn$ edges, where $m \equiv 1 (\bmod 4)$ *and* $n = 3$. Let the vertices in $C_m \times C_n$ be $\{v_1^1, v_2^1, \ldots, v_n^1; v_1^2, v_2^2, \ldots, v_n^2; \ldots, v_1^m, v_2^m, \ldots, v_n^m\}$, where v_j^i is adjacent to v_{j+1}^i, v_n^i is adjacent to $v_1^i, 1 \le i \le m, 1 \le j \le n-1$, which are called latitude cycles ; v_j^i is adjacent to v_j^{i+1}, v_j^m is adjacent to $v_j^1, 1 \le i \le m-1, 1 \le j \le n$, which are called longitude cycles. It is also called 4-regular graph. Let the vertices in $C_m \times C_n$ be $\{e_1^1, e_2^1, \ldots, e_n^1; e_1^2, e_2^2, \ldots, e_n^2; \ldots; e_1^m, e_2^m, \ldots, e_n^m\}$.

The edge labels of the latitude cycles are

$$f(v_j^i v_{j+1}^i) = 2q + 2k - 12i + 2j + 6, 1 \le i \le m, 1 \le j \le n-1$$

and $f(v_n^i v_1^i) = 2q + 2k - 12i + 2n, 1 \le i \le m$ The edge labels of the longitude cycles are

$$f(v_j^i v_j^{i+1}) = 2q + 2k - 12i + 2j - 2, 1 \le i \le m-1, 1 \le j \le n$$

and $f(v_j^m v_j^1) = 2k - 2j + 6, 1 \le j \le n.$ Now it remains to show that the *vertex labels* of *G* are all integers of the interval [0,2z].

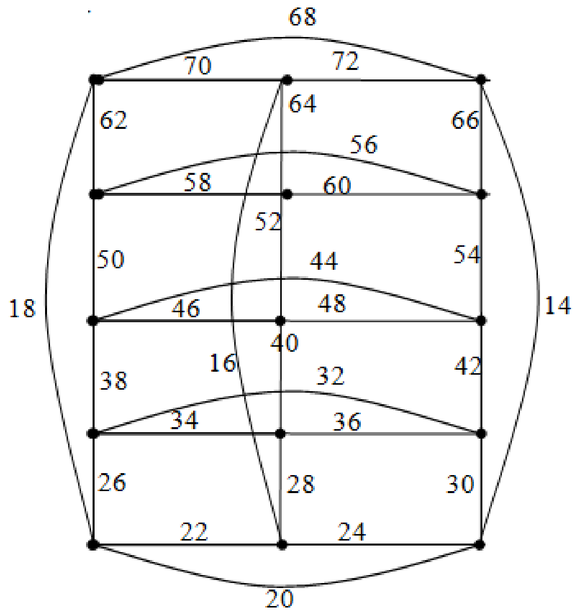
$$f(v_j^i)=\begin{cases}8k-24+2n(\bmod 2q)\ if\ i=j=1\\ 8k-28+4n(\bmod 2q)\ if\ i=1,\ j=n\\ 8k-36m+2n+24(\bmod 2q)\ if\ i=m,\ j=1\\ 8k-36m+4n+20(\bmod 2q)\ if\ i=m,\ j=n\\ 8k-36m+4n+22(\bmod 2q)\ if\ i=m,\ j=n-1\\ 8k+4n-26(\bmod 2q)\ if\ i=1,\ j=n-1\end{cases}\quad and$$

$$f(v_j^i)=\begin{cases}8k-48i+6j+2n+12(\bmod 2q)\ if\ 2\le i\le m-1,\ j=n\\ 8k-48i+6j+2n+14(\bmod 2q)\ if\ 2\le i\le m-1,\ j=1\\ 8k-48i+8j+18(\bmod 2q)\ if\ 2\le i\le m-1,\ j=n-1\end{cases}$$

$$f^*(v_j^i)=\begin{cases}2i+2(k-1)\bmod 2n(m+1)\ \ for\ \ j=1, i=1,2,\ldots,m.\\ 2(j-1)m+2(i+j-1)+2(k-1)\bmod 2n(m+1)\\ \qquad for\ \ j=2,3,\ldots,n, i=1,2,\ldots,m.\end{cases}$$

$$f^*(v_j)=\begin{cases}m(m^2+m+2k)+2(2k-1)\bmod 2n(m+1)\ \ for\ \ j=1\\ m^2(2j-1)+m(2k-5+6j)+4j+4k-6\\ for\ \ j=2,3,\ldots,n\end{cases}$$

Illustration: The 7-even even edge graceful labeling is given in Figure. 4



Definition 2.3: SF (n,m) is the graph consisting of a cycle C_n where $n \geq 3$ and n sets of m independent vertices where each set joins to each of the vertices on C_n .

Theorem 2.4: The graph S F (n,m) is k-even even edge graceful when n is odd, m is even and n divides m.

Proof: Let G be a graph SF(n,m) with $|V(G)|=|E(G)|=n(m+1)$. let $v_1, v_2, \ldots, v_n$ be vertices on the cycle of S F (n,m) and for each $j = 1, 2, \ldots, n$ the vertices $v_j^1, v_j^2, \ldots, v_j^m$ be vertices joining v_j . The edge set is the set $\{v_j v_j^i \mid i=1,2,\ldots,m \ and \ j=1,2,\ldots,n\} \cup \{v_j v_{j+1} \mid j=1,2,\ldots,n-1\} \cup \{v_n v_1\}$

Define f: E (G) $\rightarrow$ {2, 4, 6,…, 2n(m+1)} by

$$f(v_j v_j^i)=2(j-1)m+2(i+j-1)+2(k-1)$$
$$for\ i=1,2,\ldots,m\ and\ j=1,2,\ldots,n$$
$$f(v_j v_{j+1})=2j(m+1)+2(k-1)\quad for\ j=1,2,\ldots,n-1$$
$$f(v_n v_1)=2n(m+1)+2(k-1)$$

Then the induced vertex labels are as follows:

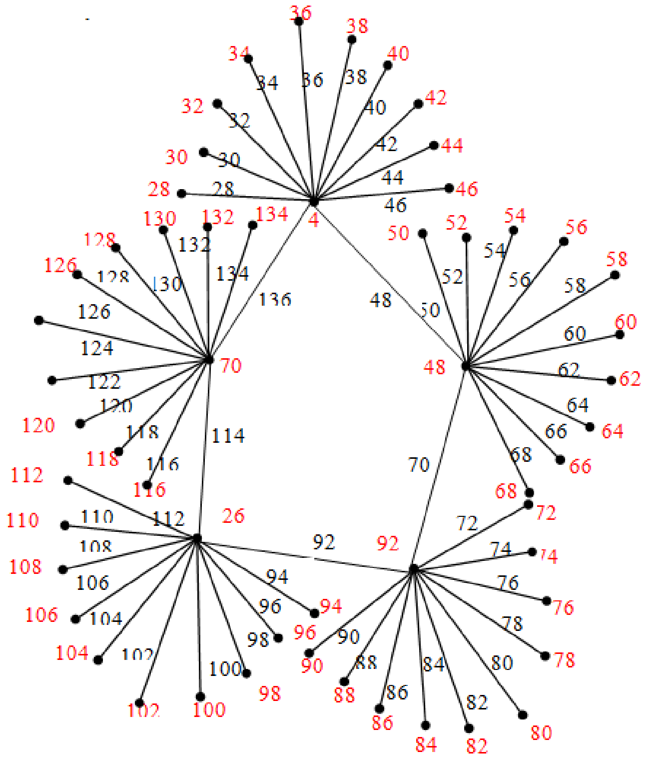


Figure 5: The 14-even even edge graceful labeling of the graph SF(5, 10).

Theorem 2.5: If f is an odd-even graceful labeling of a graph G = (V, E) with q edges then the labeling φ defined by φ (v) = (2q+2) - f (v) for all v ∈ V (G) is again an odd-even graceful labeling of G.

Proof: Let G be an odd-even graceful graph with p vertices and q edges.

Then, there exists a vertex labeling f of *G*, f: V(G) → { 1,3, …, 2q+1} and the induced function f *: E(G) → { 2,4,…, 2q} defined by f* (*e* = *uu*') = | f (*u*) - f (*u*')| ; *u, u'* ∈ *V* form an edge labeling.

Let us consider the following labeling φ of the vertices *u* of the graph *G*.
For u∈G, φ (u) = 2q+2- f (u)
Thus φ (*u*) ∈ {2q+1, 2q-1,…,1}
Further, for each edge e= uv in G,

$$\phi^*(e)=|\varphi(u)-\varphi(u')|$$
$$=|[2q+2-f(u)]-[2q+2-f(u')]|$$
$$=|f(u)-f(u')|$$

Therefore the induced edge labels are {2,4,…,2q}. Thus we get an odd-even graceful labeling φ of G.

For illustration see the graph G(2,4) in Figure. 6

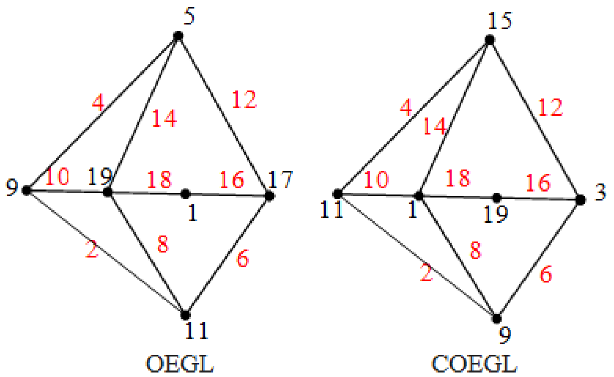


Figure 6: OEGL COEGL

Theorem 2.6: If f is an edge-odd graceful labeling of a graph G = (V, E) with q edges then the labeling φ defined by φ (e) = 2q-f (e) for all $e \in E(G)$ is again an edge-odd graceful labeling of G.

Proof: Consider the graph G (V, E) with $|V(G)| = p$ and $|E(G)| = q$.

Then *G* has a bijective edge labeling f**:** E (G) → {1,3,…, 2q-1}.

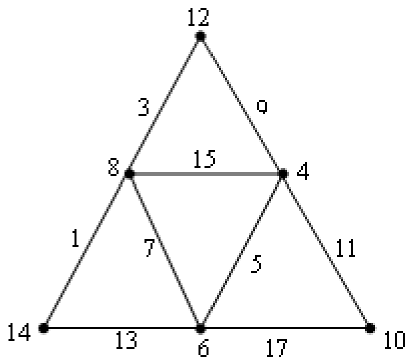
Now, define φ (e = uv) = 2q-f (e).

Then the induced function $\varphi^*(v) = \Sigma\{2q - f(uv) / uv \in E\}$ (mod 2z)

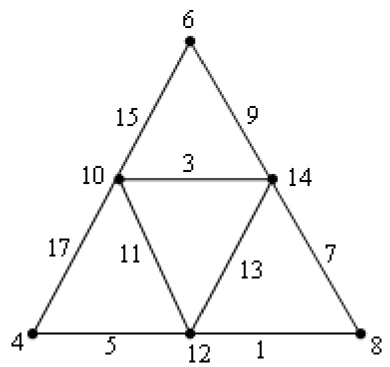
= 2q- [Σ f(e) (mod 2z)] form an vertex labeling.

Thus we get an edge-odd graceful labeling φ of G.

Illustration for edge-odd graceful and complementary edge-odd graceful graph is given in Figure. 7



edge-odd graceful graph



Complementary edge-odd graceful graph

Figure 7

3. Conclusion

In this paper, we have introduced the definitions of k-even even edge gracefulness, complementary odd-even graceful labeling and complementary edge-odd graceful labeling. We have proved that Friendship graph F_m, prism D_n, and $C_m \times C_n$ are k-even even edge graceful. Further we have proved that the graph S F (n,m) is k-even even edge graceful when n is odd, m is even and n divides m.

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