K-Even Even Edge Graceful Labeling and Some Complementary Graceful Labeling

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Abstract: S.P. Lo [4] introduced the notion of edge-graceful graphs. Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang[6] introduced the k-edge-graceful graphs. B. Gayathri, M. Duraisamy and M. Tamilselvi [3] introduced the even edge-graceful graphs. In this paper, we introduce definitions of k-even even edge graceful labeling, complementary odd-even graceful labeling, complementary edge-odd graceful labeling and we also prove that some well known graphs namely, Friendship graph $F_m$, prism $D_m C_n \times C_n$ etc., are k-even even edge graceful.

Keywords: k-even even edge graceful labeling, complementary edge-odd graceful labeling and odd-even graceful labeling.

1. Introduction

Let $G$ be a simple undirected graph with $p$ vertices and $q$ edges. Most graph labeling methods trace their origin to one introduced by Rosa [8] in 1967, or the one given by Graham and Sloane in 1980. S.P. Lo, introduced the notion of edge-graceful graphs. Sin-Min Lee, Kuo-Jye Chen and Yung-Chin Wang introduced the k-edge-graceful graphs. B. Gayathri, M. Duraisamy and M. Tamilselvi [3] introduced the even edge-graceful graphs. We have introduced a labeling called k-even even edge graceful labeling and have also introduced complementary odd-even graceful labeling and complementary edge-odd graceful labeling.

Definition 1.1: A graph is k-even even edge graceful (k>0) if there exists an injective function $f: E(G) \to \{2k,2k+2,...,2k+2q-2\}$ so that the induced map $f^*(x) = \Sigma f(xy) (mod 2z)$ where $z = \max \{p, q\}$ makes all distinct and even.

Definition 1.2: If $f$ is an odd-even graceful labeling of a graph $G = (V, E)$ with $q$ edges, then the labeling $\phi$ defined by $\phi (v) = (2q+2) - f(v)$ for all $v \in V(G)$ is again an odd-even graceful labeling of $G$ and is called complementary odd-even graceful labeling.

Definition 1.3: If $f$ is an edge-odd graceful labeling of a graph $G = (V, E)$ with $q$ edges, then the labeling $\phi$ defined by $\phi (e) = 2q- f(e)$ for all $e \in E(G)$ is again an edge-odd graceful labeling of $G$ and is called complementary edge-odd graceful labeling.

A necessary condition: If the $(p,q)$ graph $G$ is k-even even edge graceful, then $q(q+2k-1) \equiv 0 \ (mod \ z)$ where $z = \max \{p, q\}$.

Remark: 1-even even edge graceful graph is an even even edge graceful.

2. Main Results

Definition 2.1: A friendship graph $F_m$ $(m \geq 2)$ is the one point union of $m$ cycles of length 3.

Theorem 2.1.1: The Friendship graph $F_m$ is k-even even edge graceful if $m$ is odd.

Proof: Let the vertex set be $V = \{v_i | 1 \leq i \leq 2m\}$ and the edge set be $E = \{e_{ij} = v_i v_{j+1} | j \text{ is odd and } i \equiv 0 (mod 3)\} \cup \{e_{ij} = v_i v_j | 1 \leq j \leq 2m \text{ and } i \equiv 1,2 (mod 3)\}$. Clearly $|V| = 2m+1$ and $|E| = 3m$.

Define $f: E(G) \to \{2k,2k+2,...,2k+2q-2\}$ as follows: Case (1): $k \equiv 0 \ (mod \ 3)$

Let $f(e_1) = 2k+6m-4$, $f(e_2) = 2k+6m-2$, $f(e_3) = 2k$ and $f(e_4) = 2k+(2i-6)$ ; $i = 4,5,...,2m$.

Case (2): $k \equiv 1 \ (mod \ 3)$

Let $f(e_1) = 2k$, $f(e_2) = 2k+2$, $f(e_3) = 2k+4$ and $f(e_4) = 2k+(2i-2)$ ; $i = 4,5,...,2m$.

Case (3): $k \equiv 2 \ (mod \ 3)$

Let $f(e_1) = 2k+6m-2$, $f(e_2) = 2k$, $f(e_3) = 2k+2$ and $f(e_4) = 2k+(2i-4)$ ; $i = 4,5,...,2m$.

Thus the induced vertex labels are: Case (1): $k \equiv 0 \ (mod \ 3)$

Let $f(v_1) = 4k+6j-10 \ (mod \ 6m)$ $i = 3,5,...,2m-1$;

$f(v_2) = 4k+6j-14 \ (mod \ 6m)$ $i = 2,4,...,2m$ & $f(v) = 0$.

Case (2): $k \equiv 1,2 \ (mod \ 3)$

Let $f(v_1) = 4k+6j-2 \ (mod \ 6m)$ $i = 3,5,...,2m-1$;

$f(v_2) = 4k+6j-6 \ (mod \ 6m)$ $i = 2,4,...,2m$ & $f(v) = 0$.

Figure 1: 6-even even edge gracefully labeled Friendship graph $F_6$.
Theorem 2.2.1: Prism $D_n$ is k-even even edge graceful.

Proof: Let $G$ be a Prism graph with $2n$ vertices and $3n$ edges. Let $\{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ be the set of vertices and edges

\[
\begin{align*}
\{v_nv_1 & \quad \text{for } i = 1 \\
v_{i-1}v_i & \quad \text{for } 2 \leq i \leq n \text{ and } n+2 \leq i \leq 2n \\
v_{2n}v_{n+1} & \quad \text{for } i = n+1 \\
v_{n+1}v_1 & \quad \text{for } i = 2n+1 \\
v_{2n-j}v_{n-j} & \quad \text{for } 2n+2 \leq i \leq 3n \text{ and } 0 \leq j \leq n-2
\end{align*}
\]

Illustration: Figure 3 shows 14 -even even edge graceful labeling of Prism $D_n$.

First, we label the edges as follows:
Define $f(e_i) = 2k+2i$ for $1 \leq i \leq 2n$.
Then the induced vertex labels are as follows:
Case (1): $f(v_i) = f(e_i) + f(e_{i+1}) + f(e_{2n+1})$
$\equiv 2k+2+2k+2+2k+2(2n+1) \equiv 0 (\text{mod } 6n)$
Case (2): $f(v_i) = f(e_i) + f(e_{i+1}) + f(e_{3n+2})$ for $2 \leq i \leq n-1$
$\equiv 2k+2i+2k+i+1+2k+2(3n+2) \equiv 0 (\text{mod } 6n)$
$\equiv 6k+2i+4 (\text{mod } 6n)$
Case (3): $f(v_i) = f(e_i) + f(e_{n+1}) + f(e_{2n})$
$\equiv 2k+2n-2+2k+2+2k+2(2n)-2 \equiv 0 (\text{mod } 6n)$
$\equiv 6k+2n+2 (\text{mod } 6n)$
Case (4): $f(v_i) = f(e_{n+1}) + f(e_{2n+2})$
$\equiv 2k+2n+2-2+2k+2(2n+2) \equiv 0 (\text{mod } 6n)$
$\equiv 6k+n (\text{mod } 6n)$
Case (5): $f(v_i) = f(e_{n+1}) + f(e_{2n+2})$
$\equiv 2k+2n+2-2+2k+2(2n+2) \equiv 0 (\text{mod } 6n)$
$\equiv 6k+4n+2 (\text{mod } 6n)$

Hence induced vertex labels of the graph are
$\{f(v_1), f(v_2), \ldots, f(v_{n+1}), f(v_{n+2}), \ldots, f(v_{2n})\}$, where $f(v_j)$ is adjacent to $f(v_{j+1})$.

Theorem 2.3: The graph $C_m \times C_n$ is k-even even edge graceful.

Proof: Let the web graph $C_m \times C_n$ be a graph with $mn$ vertices and $2mn$ edges, where $m \equiv 1 (\text{mod } 4)$ and $n = 3$. Let the vertices in $C_m \times C_n$ be

\[
\{v_1, v_2, \ldots, v_m, v_{m+1}, v_{m+2}, \ldots, v_{m+3}, v_{m+4}, \ldots, v_{m+n}, v_{m+n+1}, v_{m+n+2}, \ldots, v_{m+n+3}\}
\]

The edge labels of the latitude cycles are
$f(v_j, v_{j+1}) = 2q + 2k - 12i + 2j + 6$, $1 \leq i \leq m, 1 \leq j \leq n-1$
and $f(v_j, v_{j+1}) = 2q + 2k - 12i + 2j + 6$, $1 \leq i \leq m$.
The edge labels of the longitude cycles are
$f(v_j, v_{j+1}) = 2q + 2k - 12i + 2j - 2$, $1 \leq i \leq m-1, 1 \leq j \leq n$
and $f(v_j, v_{j+1}) = 2q + 2k - 12i + 2j - 2$, $1 \leq i \leq m-1, 1 \leq j \leq n$. Now it remains to show that the vertex labels of $G$ are all integers of the interval $[0, 2z]$. 

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Theorem 2.4: The graph S F (n,m) is k-even even edge graceful if n is odd, m is even and n divides m.

Proof: Let G be a graph SF(n,m) with |V(G)| = |E(G)| = n(m+1). Let v₁, v₂, ..., vₙ be vertices on the cycle of S F (n,m) and for each j = 1, 2, ..., n the vertices vⱼ, vⱼ', ..., vₙ are vertices joining vⱼ. The edge set is the set \( \{v_j,v'_j \mid i=1,2,\ldots,m\text{ and } j=1,2,\ldots,n\} \cup \{v_j,v_{j+1} \mid j=1,2,\ldots,n-1\} \cup \{v_n,v_1\} \).

Define \( f: E(G) \rightarrow \{2,4,6,\ldots,2n(m+1)\} \) by

\[
f(v_j,v'_j) = 2(j-1)m + 2(i + j - 1) + 2(k-1) \quad \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n.
\]

\[
f(v_j,v_{j+1}) = 2(jm + 1) + 2(k-1) \quad \text{for } j = 1,2,\ldots,n-1.
\]

\[
f(v_n,v_1) = 2n(m+1) + 2(k-1)
\]

Then the induced vertex labels are as follows:

\[
f^*(v_j) = \begin{cases} 2i + 2(k-1) \mod 2n(m+1) & \text{for } j = 1, i = 1,\ldots,m. \\ 2(j-1)m + 2(i + j - 1) + 2(k-1) \mod 2n(m+1) & \text{for } j = 2,3,\ldots,n, i = 1,\ldots,m. \end{cases}
\]

Illustration: The 7-even even edge graceful labeling is given in Figure. 4.

Figure 5: The 14-even even edge graceful labeling of the graph SF(5, 10).

Theorem 2.5: If \( f \) is an odd-even graceful labeling of a graph \( G = (V,E) \) with \( q \) edges then the labeling \( \phi \) defined by \( \phi(v) = (2q+2) - f(v) \) for all \( v \in V(G) \) is again an odd-even graceful labeling of \( G \).

Proof: Let \( G \) be an odd-even graceful graph with \( p \) vertices and \( q \) edges.

Then, there exists a vertex labeling \( f \) of \( G \), \( f: V(G) \rightarrow \{1,3,\ldots,2q+1\} \) and the induced function \( f^*: E(G) \rightarrow \{2,4,\ldots,2q\} \) defined by \( f^*(e = uu') = |f(u) - f(u')| \); \( u, u' \in V \) form an edge labeling.

Let us consider the following labeling \( \phi \) of the vertices \( u \) of the graph \( G \).

For \( u \in G \), \( \phi(u) = 2q+2 - f(u) \)

Thus \( \phi(u) \in \{2q+1,2q,\ldots,1\} \). Further, for each edge \( e = uv \in G \),

\[
\phi^*(e) = |\phi(u) - \phi(v')|
\]

\[
= \begin{cases} 2q+2 - f(u) & \text{if } e = uu' \in E(G) \\ f(u) - f(u') & \text{otherwise}. \end{cases}
\]

Therefore the induced edge labels are \{2,4,\ldots,2q\}. Thus we get an odd-even graceful labeling \( \phi \) of \( G \).

For illustration see the graph \( G(2,4) \) in Figure. 6.
Theorem 2.6: If \( f \) is an edge-odd graceful labeling of a graph \( G = (V, E) \) with \( q \) edges then the labeling \( \phi \) defined by \( \phi(e) = 2q - f(e) \) for all \( e \in E(G) \) is again an edge-odd graceful labeling of \( G \).

Proof: Consider the graph \( G = (V, E) \) with \(|V(G)| = p \) and \(|E(G)| = q\).

Then \( G \) has a bijective edge labeling \( f: E(G) \rightarrow \{1, 3, \ldots, 2q-1\} \).

Now, define \( \phi(e = uv) = 2q - f(e) \).

Then the induced function \( \phi^* : \{v\} \rightarrow \{2q - \sum_{\text{uv} \in E} f(e) \mod 2x\} \) form an vertex labeling.

Thus we get an edge-odd graceful labeling \( \phi \) of \( G \).

Illustration for edge-odd graceful and complementary edge-odd graceful graph is given in Figure 7.

3. Conclusion

In this paper, we have introduced the definitions of k-even even edge gracefulness, complementary odd-even graceful labeling and complementary edge-odd graceful labeling. We have proved that Friendship graph \( F_n \), prism \( D_n \), and \( C_m \times C_n \) are k-even even edge graceful. Further we have proved that the graph \( S F(n,m) \) is k-even even edge graceful when \( n \) is odd, \( m \) is even and \( n \) divides \( m \).

References


