Constrain Control of Trigonometric Rational Quadratic Spline with Shape Parameters Based on Function Values

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Abstract: ***The aim of this paper presents an analysis of weighted quadratic trigonometric spline with two shape parameters which interpolate on function value. This interpolating is C^1 continuous with quadratic denominator. Constrain control of this rational quadratic trigonometric spline is derived which force it be bound in the given region. Approximation property is discussed.***

Keywords: trigonometric spline; shape parameter; interpolation; constrain control; approximation.

1. Introduction

Recently curve designing is playing a very important role. The study of curve and its analysis is very important in the field of CAGD. In electronics importance of trigonometric rational spline is surprising. In CAGD, it is often necessary to generate curves and surfaces that approximate shapes with some desired shape features. Shape parameters are playing a very important role [2,3,4,5] in these applications. In many applications of geometric modeling, it is often to keep the curve bounded in the given region. In this paper constrain for the spline by using the suitable parameter which force it to be in the given region is analyse and derived. Sometimes it is difficult to obtain derivatives to overcome this problem this paper is based on the use of function values only.

The paper is arranged as follows. In Section 2, rational trigonometric quadratic spline is constructed by using shape parameters. In section 3, condition for the interpolating curve is bounded between two lines is derived. In section 4, a numerical example with graphical representation is given. Section 5, is about the approximation properties of such trigonometric spline. Conclusion of the paper is discussed in section 6.

2. Trigonometric Spline Interpolation

In this section, the quadratic trigonometric function is defined as, let $t_0 < t_1 < \dots\dots\dots < t_n < t_{n+1}$ are the knots f_i for $i = 0,1,2,3 \dots\dots, n, n+1$ are the function values defined at the knots. Let $\theta_i = \frac{(t-t_i)}{h_i}$ for $t \in [t_i, t_{i+1}]$ and let αi and βi are positive parameters and $\alpha \in (0,1]$.

Now consider,

$$p_i(t) = \alpha_i\left(1-\sin\frac{\Pi\theta}{2}\right)\left[1+(1-\alpha)\sin\frac{\Pi\theta}{2}\right]A_i + \alpha\sin\frac{\Pi\theta}{2}\left(1-\sin\frac{\Pi\theta}{2}\right)B_i + \alpha\cos\frac{\Pi\theta}{2}\left(1-\cos\frac{\Pi\theta}{2}\right)C_i + \beta_i\left(1-\cos\frac{\Pi\theta}{2}\right)\left[1+(1-\alpha)\cos\frac{\Pi\theta}{2}\right]D_i$$

$$q_i(t) = \alpha_i\left(1-\sin\frac{\Pi\theta}{2}\right)\left[1+(1-\alpha)\sin\frac{\Pi\theta}{2}\right] + \alpha\sin\frac{\Pi\theta}{2}\left(1-\sin\frac{\Pi\theta}{2}\right) + \alpha\cos\frac{\Pi\theta}{2}\left(1-\cos\frac{\Pi\theta}{2}\right) + \beta_i\left(1-\cos\frac{\Pi\theta}{2}\right)\left[1+(1-\alpha)\cos\frac{\Pi\theta}{2}\right]$$

$$P(t) = p_i(t)/q_i(t) \quad (1)$$

which satisfying the following interpolation conditions

$$p_i(t_i) = f_i \text{ and } p'_i(t_i) = \Delta_i, \quad (2)$$

for $i = 0,1,2,3 \dots\dots, n, n-1$

Thus we have,

$$A_i = f_i, D_i = f_{i+1}$$

$$B_i = f_i + \frac{2h_i\alpha_i\Delta_i}{\alpha\pi}$$

and

$$C_i = f_{i+1} - \frac{2h_i\beta_i\Delta_{i+1}}{\alpha\pi}$$

3. Constrained interpolation of rational quadratic trigonometric spline

To get the condition for the rational quadratic function defined in (1) to bounded between lines in the interpolating interval we shall consider the conditions $t_0 < t_1 < \dots < t_n < t_{n+1}$ and a data set $(t_i, f_i): i = 0,1,\dots, n, n+1$ a straight line g(t) is defined as $f_i \geq (\leq)(\leq)g(t_i), i = 0,1,\dots, n, n+1$ for all t of given interval.

Consider two cases:

first for "above" line $P(t) \geq g(t)$ consider subinterval $[t_i, t_{i+1}]$ in which $q_i \geq 0$,

$$P(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

gives,

$$R_i(t) \geq 0$$

where

$$R_i(t) = p_i(t) - q_i(t)g(t)$$

Then,

$$R_i(t) = \left[\left(\alpha_i\left(1 - sin\frac{\Pi\theta}{2}\right)\left[1 + (1-\alpha)sin\frac{\Pi\theta}{2}\right]\right]U_i + \alpha sin\frac{\Pi\theta}{2}\left(1 - sin\frac{\Pi\theta}{2}\right)\left(U_i + \frac{2\alpha_i h_i}{\alpha\pi}\Delta_i\right) + \alpha\, cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right)\left(V_i + \frac{2\beta_i h_i}{\alpha\pi}\Delta_{i+1}\right) + \left[\beta_i\left(1 - cos\frac{\Pi\theta}{2}\right)\left[1 + (1-\alpha)cos\frac{\Pi\theta}{2}\right]\right]V_i$$

g_i, g_{i+1} represent $g_i(t), g_{i+1}(t)$ respectively

Where

$$\boldsymbol{U_i = (1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i \geq 0}$$

$$V_i = (1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i \geq 0$$

Theorem 1:

Given, $(t_i, f_i, g_i), i = 0,1,2, \ldots\ldots, n, n+1$, with $g_i \leq f_i$ and $\alpha_i, \beta_i > 0$ and $\alpha \in$ (0,1] the sufficient condition for the interpolation defined by (1) lies above the line g(t) in $[t_i, t_{i+1}]$ is that

$$U_i = (1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i \geq 0$$

$$V_i = (1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + \frac{2h_i\alpha_i}{\propto \Pi}\Delta_i \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i - \frac{2h_i\beta_i}{\alpha\Pi}\Delta_{i+1} \geq 0 \quad (2)$$

For the equally spaced knots to keep the interpolating curve lies above line see the following corollary:

Corollary 1 For uniform spaced partition, the sufficient condition for the rational quadratic trigonometric curve P (t) defined in (1) lies above the line g(t) when α_i, β_i positive and $\alpha \in (0,1]$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) \geq 0 \quad (3)$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) + (f_{i+1} - f_i) \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) + \frac{2\alpha_i}{\alpha\pi}(f_{i+1} - f_i) \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta(f_{i+1} - f_i) + f_{i+1} - f_i - \frac{2\beta_i}{\alpha\pi}(f_{i+2} - f_{i+1}) \geq 0$$

Similarly, the sufficient conditions for the rational quadratic trigonometric curve P(t) defined in (1) lies below the line g*(t) gives the following theorem:

Theorem2

Given $(t_i, f_i, g_i, g_i^*), i = 0,1,2, \ldots\ldots, n, n+1$, with $g_i \leq f_i \leq g_i^*$ and $\alpha_i, \beta_i > 0$ and $\alpha \in$ (0,1] the sufficient condition for the interpolation defined by (1) lies above the line g(t) and below the line g_i^* in $[t_i, t_{i+1}]$ is that

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i \geq 0 \quad (4)$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + \frac{2\alpha_i h_i}{\alpha\pi}\Delta_i \geq 0$$

$$(1-\theta)(f_i - g_i) + \theta(f_{i+1} - g_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i - \frac{2\beta_i h_i}{\alpha\pi}\Delta_{i+1} \geq 0$$

and

$$(1-\theta)(g_i^* - f_i) + \theta(g_{i+1}^* - f_{i+1}) - \theta h_i \Delta_i \geq 0$$

$$(1-\theta)(g_i^* - f_i) + \theta(g_{i+1}^* - f_{i+1}) - \theta h_i \Delta_i + h_i \Delta_i \geq 0$$

$$(1-\theta)(g_i^* - f_i) + \theta(g_{i+1}^* - f_{i+1}) + \theta h_i \Delta_i - \frac{2\alpha_i h_i}{\alpha\Pi}\Delta_i \geq 0$$

$$(1-\theta)(g_i^* - f_i) + \theta(g_{i+1}^* - f_{i+1}) + \theta h_i \Delta_i - h_i \Delta_i + \frac{2\beta_i h_i}{\alpha\Pi}\Delta_{i+1} \geq 0 \quad (4)$$

Numerical Example

The rational cubic trigonometric curve P(t) define in (1) is bounded between $g(t)$ and $g^*(t)$ for this data is given in the following table, which satisfied constrain inequalities proved in theorem.

I	t_i	$g^*(t_i)$	$f(t_i)$	$g(t_i)$	α_i	β_i
1	0.0	0.107	0.1	0.093	0.001123	0.0011423
2	0.5	0.027	0.02	0.013	0.001555	0.00124
3	1.0	0.087	0.08	0.073	0.001510	0.0011905
4	1.5	0.047	0.04	0.033		
5	2.0	0.067	0.06	0.053		

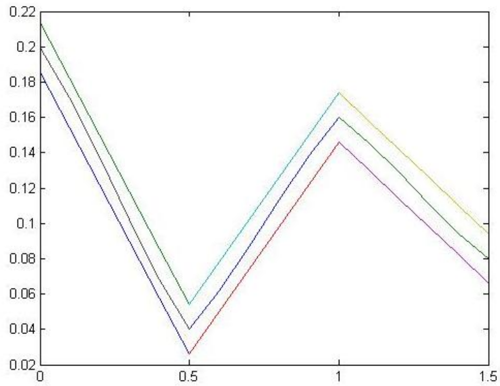

Figure 1: Graph of $g(t)$, $g^*(t)$, $P(t)$ function.

4. Approximation Properties of the Weighted Interpolation

To find the error estimation we consider that the given in $f(g) \in C^2$ and p(t) is the interpolating function of $f(t)$ in $[t_i, t_{i+1}]$ for $i = 0,1, \ldots\ldots, n+1$, consider the case that the knots are equally spaced, namely, $h_i = h = \frac{t_n - t_0}{n}$ for all $i = 1,2, \ldots\ldots, n$, using the Peano-Kernel Theorem [13] gives the following

$R[F] = f(t) - p(t) = \int f^2(\tau)R_t[(t - \tau)_+]d\tau, t \in [t_i, t_{i+1}]$,

where,

$$R_t[(t-\tau)_+] = = \begin{cases} p(\tau), t_i < \tau < t \\ q(\tau), t < \tau < t_{i+1} \\ r(\tau), t_{i+1} < \tau < t_{i+2} \end{cases}$$

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^{(2)}(t)\| \left[\int_{t_i}^{t} |p(\tau)|\, dr + \int_{t}^{t_{i+1}} |q(\tau)|\, dr + \int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\, dr\right] \quad (5)$$

Next, we calculate the following terms in (5)

$$r(\tau) = \frac{2\beta_i h_i}{K\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right)$$

$$\int_{t_{i+1}}^{t_{i+2}} |r(\tau)|\, dr = \frac{2h_i^2\beta_i}{K\pi} cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right) = h_i^2 W_1$$

$$K = \alpha_i\left(1 - sin\frac{\Pi\theta}{2}\right)\left[1 + (1-\alpha) sin\frac{\Pi\theta}{2}\right] + \alpha sin\frac{\Pi\theta}{2}\left(1 - sin\frac{\Pi\theta}{2}\right) + \alpha\, cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right) + \beta_i\left(1 - cos\frac{\Pi\theta}{2}\right)\left[1 + (1-\alpha) cos\frac{\Pi\theta}{2}\right]$$

$$N = \left(\frac{2\alpha_i}{\pi} sin\frac{\Pi\theta}{2}\left(1 - sin\frac{\Pi\theta}{2}\right) + \left(\alpha - \frac{2\beta_i}{\pi}\right) cos\frac{\pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right) + \beta_i\left(1 - cos\frac{\Pi\theta}{2}\right)\left[1 + (1-\alpha) cos\frac{\Pi\theta}{2}\right]\right)$$

We observe that for

$$q(t_{i+1}) = \frac{2\beta_i h_i}{K\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right) \geq 0$$

$$q(t) = -\frac{h_i}{K}\left(N(1-\theta) - \left[\frac{2\beta_i}{\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right)\right]\right) \leq 0$$

and

so there exist root

$$\tau_1^* = t_{i+1} - \frac{2\beta_i h_i}{N\Pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right)$$

Thus,

$$\int_t^{t_{i+1}} |q(\tau)|\, dr = \int_t^{\tau_1^*} -q(\tau)\, dr + \int_{\tau_1^*}^{t_{i+1}} -q(\tau)\, dr$$

$$\int_t^{\tau_1^*} -q(\tau)\, dr + \int_{r^*}^{t_{i+1}} q(\tau)\, dr = h_i^2(P - Q) = h_i^2 W_2$$

$$P = \frac{N}{K}\left[\frac{(1-\theta)^2}{2} - \left(\frac{2\beta_i h_i}{\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right)\right)^2\right]$$

$$Q = \frac{2\beta_i h_i}{K\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right)\left[(1-\theta) - \frac{4\beta_i}{K\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right)\right]$$

$$\int_t^{t_{i+1}} |q(\tau)|\, dr = h_i^2 W_2$$

Similarly, since

$$p(\tau) = (t - \tau) + q(\tau)$$

$$p(t) = q(t) \leq 0, p(t_i) = (t - t_i) + q(t_i)$$

$$p(t_i) = (t - t_i) + \frac{1}{K} h_i N - \frac{2\beta_i}{\pi} cos\frac{\pi\theta}{2}\left(1 - cos\frac{\pi\theta}{2}\right) > 0$$

there is a root

$$\tau_2^* = t_{i+1} - \frac{K h_i}{K - N}(1-\theta) - \frac{2\beta_i}{K\pi} cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right)$$

Let

$$L = \frac{2\beta_i}{K\pi} cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right)$$

$$\int_t^{t_{i+1}} |p(\tau)|\, dr = \int_t^{\tau_2^*} -p(\tau)\, dr + \int_{\tau_2^*}^{t_{i+1}} p(\tau)\, dr = h_i^2 W_3$$

where,

$$W_3 = z_1 + z_2 + z_3$$

$$z_1 = (1-\theta) - \frac{K h_i}{K - N}\left((1-\theta) - L\right)^2 - \theta^2$$

$$z_2 = -\frac{1}{2}\left[cos^2\frac{\Pi\theta}{2} + \frac{2}{\Pi}(1-\lambda)\left(1 - sin\frac{\Pi\theta}{2}\right)\right]\left[\theta^2 - \left(\frac{P}{Q}\right)^2\right]$$

$$z_3 = -\frac{2}{\Pi}\left[cos\frac{\Pi\theta}{2}\left(1 - cos\frac{\Pi\theta}{2}\right)\right]\left[-(1-\theta) + \left(\frac{P}{Q}\right)\right]$$

from the above calculations it can be shown that

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^{(2)}(t)\| h^2 M \quad (6)$$

Where $M = W_1 + W_2 + W_3$ is depending upon θ

5. Conclusion

In this paper, a quadratic rational trigonometric spline was constructed. The conditions was derived for spline function forcefully to be in the given region which is useful in the field of engineering and surface designing.

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