If N is a contraction, i.e.

$$\|N(x) - N(y)\| \leq k\|x - y\|, 0 < k < 1,$$

Then,

$$\|y_{m+1}\| = \|N(y_0 + y_1 + \cdots + y_m) - N(y_0 + y_1 + \cdots + y_{m-1})\|$$

$$\leq k\|y_m\| \leq \cdots \leq k^n\|y_0\|, \ m = 0,1,2, \ldots,$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to solution of Eq. (1) [18], which is unique, in view of Banach fixed point theorem[19]. The k-term approximate solution of Eq. (2) and (3) is given by $\sum_{i=0}^{k-1} y_i$.

3. New Iterative Method for the System

For simplicity, let us rewrite the system of nonlinear Volterra integral equations in Eq. (1) above in vector form as:

$$y^{(m)}(x) = f(x) + \int_0^x (K(x,t,y_1^{(m)}, y_2^{(m)}, \ldots, y_n^{(m)})\, dt \qquad (9)$$

where

$$f(x) = [f_1(x),\ f_2(x), \ldots, f_n(x)]^T,$$

$$y^{(m)}(x) = [y_1^{(m)}, y_2^{(m)}, \ldots, y_n^{(m)}]^T,$$

$$\mathbf{K} = [K_1, K_2, \ldots, K_n]$$

In view of the new iterative method, the system of Volterra integro-differential equation in Eq. (9) is equivalent to the system of integral equation:

$$y(x) = (x) + I_x^m(\int_0^x (K(x,t,y_1^{(m)}, y_2^{(m)}, \ldots, y_n^{(m)})\, dt) \qquad (10)$$

$$= y_0(x) - N(y) \qquad (11)$$

where the zero solution $y_0(x)$ is the solution of the system of nth-order integro-differential equation and

$$\frac{d^n y_0}{dx^n} = g(x) - f(x)y(x), \frac{d^n y_0}{dx^n} = \alpha_r, \ r = 0,1,2, \ldots, n-1. \qquad (12)$$

$$y_0(x) = [y_{10}(x),\ y_{20}(x), \ldots, y_{n0}(x)]^T,$$

$$y(x) = [y_1(x), y_2(x), \ldots, y_n(x)]^T$$

$$f(x) = [f_1(x),\ f_2(x), \ldots, f_n(x)]^T,$$

$$\mathbf{K} = [K_1, K_2, \ldots, K_n]$$

Therefore,

$$N(y) = I_x^n\left(\int_0^x \left(K(x,t) y^{(q)}(t) y^{(m)}(t)\right) dt\right) \qquad (13)$$

where I_x^n is an nth-order integral operator with respect to x.

In vector notation, the new iterative algorithm for (13) is:

$$y_1(x) = I_x^n\left(\int_0^x \left(K(x,t) y_{10}{}^{(q)}(t) y_{20}{}^{(m)}(t)\right) dt\right)$$

$$y_2(x) = I_x^n(\int_0^x (K(x,t)(y_{10}{}^{(q)}(t) + y_{11}{}^{(q)}(t)(y_{20}{}^{(m)}(t) + y_{21}{}^{(q)}(t))) dt) - I_x^n\left(\int_0^x \left(K(x,t) y_{10}{}^{(q)}(t) y_{20}{}^{(m)}(t)\right) dt\right)$$

$\cdots$

$$y_r(x) = I_x^n(\int_0^x (K(x,t)(y_{10}{}^{(q)}(t) + \cdots + y_{1,r}{}^{(q)}(t))(y_{20}{}^{(m)}(t) + \cdots + y_{2,r}{}^{(q)}(t)) dt) - I_x^n(\int_0^x (K(x,t)(y_{10}{}^{(q)}(t) + \cdots + y_{1,r-1}{}^{(q)}(t))(y_{20}{}^{(m)}(t) + \cdots + y_{2,r-1}{}^{(q)}(t)) dt)$$

where:

$$y_1 = [y_{11}, y_{21}, \ldots, y_{n1}]^T,$$

$\cdots$

$$y_r = [y_{1r}, y_{2r}, \ldots, y_{nr}]^T, \text{ and}$$

$$K = [K_1, K_2, \ldots, K_n]^T,$$

4. Illustrative Examples

Example 1.

Consider the system of first-order linear Volterra integro-differential equation:

$$y_1'(x) = 1 + x - \frac{1}{3}x^3 + \int_0^x ((x-t)y_1(t) + (x-t)y_2(t))\, dt \qquad (14)$$

$$y_2'(x) = 1 - x - \frac{1}{12}x^4 + \int_0^x ((x-t)y_1(t) - (x-t)y_2(t))\, dt$$

with the initial conditions:

$$y_1(0) = 0,\ y_1'(0) = 0 \text{ and}$$

$$y_2(0) = 0,\ y_2'(0) = 0$$

The system of the integro-differential equation (14) is equivalent to the system of integral equation:

$$y_{10}(x) = x + \frac{1}{2}x^2 - \frac{1}{12}x^4 + I_x'[\int_0^x ((x-t)y_1(t) + (x-t)y_2(t)) dt]$$

$$y_{20}(x) = x - \frac{1}{2}x^2 - \frac{1}{60}x^5 + I_x'[\int_0^x ((x-t)y_1(t) - (x-t)y_2(t)) dt]$$

Let

$$N_1(y) = I_x'[\int_0^x ((x-t)y_1(t) + (x-t)y_2(t)) dt]$$

$$N_2(y) = I_x'[\int_0^x ((x-t)y_1(t) - (x-t)y_2(t)) dt]$$

we obtain easily the following first few components of the new iterative method solution.

The first five terms are:

$$y_{10}(x) = x + \frac{1}{2}x^2 - \frac{1}{12}x^4$$

$$y_{20}(x) = x - \frac{1}{2}x^2 - \frac{1}{60}x^5$$

$$y_{11}(x) = I_x'[\int_0^x ((x-t)y_{10}(t) + (x-t)y_{20}(t)) dt] = -\frac{1}{20160}x^8 - \frac{1}{2520}x^7 + \frac{1}{12}x^4$$

$$y_{21}(x) = I_x'[\int_0^x ((x-t)y_{10}(t) - (x-t)y_{20}(t)) dt] = \frac{1}{20160}x^8 - \frac{1}{2520}x^7 + \frac{1}{60}x^5$$

$$y_{12}(x) = I_x'[\int_0^x ((x-t)(y_{10}(t) + y_{11}(t)) + (x-t)(y_{20}(t) + y_{21}(t))) dt] - I_x'[\int_0^x ((x-t)y_{10}(t) + (x-t)y_{20}(t)) dt]$$

$$= -\frac{1}{907200}x^{10} + \frac{1}{20160}x^8 + \frac{1}{2520}x^7$$

$$y_{22}(x) = I_x'\left[\int_0^x \left((x-t)(y_{10}(t)+y_{11}(t)) - (x-t)(y_{20}(t)+y_{21}(t))\right)dt\right] - I_x'\left[\int_0^x \left((x-t)y_{10}(t) - (x-t)y_{20}(t)\right)dt\right]$$

$$= -\frac{1}{9979200}x^{11} - \frac{1}{20160}x^8 + \frac{1}{2520}x^7$$

$$y_{13}(x) = -\frac{1}{21794572800}x^{14} - \frac{1}{1556755200}x^{13} + \frac{1}{907200}x^{10}$$

$$y_{23}(x) = \frac{1}{21794572800}x^{14} - \frac{1}{1556755200}x^{13} + \frac{1}{9979200}x^{11}$$

$$y_{14}(x) = -\frac{1}{2615348736000}x^{16} + \frac{1}{21794572800}x^{14} + \frac{1}{1556755200}x^{13}.$$

$$y_{24}(x) = -\frac{1}{44460928512000}x^{17} - \frac{1}{21794572800}x^{14} + \frac{1}{1556755200}x^{13}.$$

and so on, in the same manner, the rest components can be obtained. The sum of the first five terms is:

$$y_1(x) = x + \frac{1}{2}x^2 - \frac{1}{2615348736000}x^{16}$$

$$y_2(x) = x - \frac{1}{2}x^2 - \frac{1}{44460928512000}x^{17}$$

It is obvious that the iterations converge to the exact solutions $y_1(x) = x + \frac{1}{2}x^2$ and $y_2(x) = x - \frac{1}{2}x^2$ as the number of iterations becomes large.

In Figure 1, we have plotted $y_1(x) = \sum_{m=0}^{4} y_{1m}(x)$ and $y_2(x) = \sum_{m=0}^{4} y_{2m}(x)$.

Example 2

Consider the system of second-order linear integro-differential equations

$$y_1''(x) = -x^3 + x^4 + \int_0^x \left(3y_2(t) + 4y_3(t)\right)dt,$$

$$y_2''(x) = 2 + x^2 - x^4 + \int_0^x \left(4y_3(t) - 2y_1(t)\right)dt, \quad (15)$$

$$y_3''(x) = 6x - x^2 + x^3 + \int_0^x \left(2y_1(t) - 3y_2(t)\right)dt$$

with the initial conditions:

$y_1(0) = 0,\ y_1'(0) = 1.$

$y_2(0) = 0,\ y_2'(0) = 0.$

$y_3(0) = 0,\ y_3'(0) = 0.$

The system of integro-differential equation is equivalent to the system of integral equation:

$$y_{10}(x) = x - \frac{1}{20}x^5 - \frac{1}{30}x^6 + I_x^2\left[\int_0^x \left(3y_2(t) + 4y_3(t)\right)dt\right]$$

$$y_{20}(x) = x^2 + \frac{1}{12}x^4 - \frac{1}{30}x^6 + I_x^2\left[\int_0^x \left(4y_3(t) - 2y_1(t)\right)dt\right]$$

$$y_{30}(x) = x^3 - \frac{1}{12}x^4 + \frac{1}{20}x^5 + I_x^2\left[\int_0^x \left(2y_1(t) - 3y_2(t)\right)dt\right]$$

Let $N_1(y) = I_x^2\left[\int_0^x \left(3y_2(t) + 4y_3(t)\right)dt\right]$,

$N_2(y) = I_x^2\left[\int_0^x \left(4y_3(t) - 2y_1(t)\right)dt\right]$ and

$N_3(y) = I_x^2\left[\int_0^x \left(2y_1(t) - 3y_2(t)\right)dt\right]$

we obtain the first few components of the new iterative method solution.

The first four terms are:

$$y_{10}(x) = x - \frac{1}{20}x^5 - \frac{1}{30}x^6$$

$$y_{20}(x) = x^2 + \frac{1}{12}x^4 - \frac{1}{30}x^6$$

$$y_{30}(x) = x^3 - \frac{1}{12}x^4 + \frac{1}{20}x^5$$

$$y_{11}(x) = \frac{1}{20}x^5 - \frac{1}{2520}x^7 - \frac{1}{5040}x^9 + \frac{1}{30}x^6 + \frac{1}{1680}x^8$$

$$y_{21}(x) = \frac{1}{30}x^6 - \frac{1}{630}x^7 + \frac{1}{1120}x^8 - \frac{1}{12}x^4 + \frac{1}{7560}x^9$$

$$y_{31}(x) = \frac{1}{12}x^4 - \frac{1}{3360}x^8 + \frac{1}{15120}x^9 - \frac{1}{20}x^5 - \frac{1}{840}x^7$$

$$y_{12}(x) = -\frac{1}{75600}x^{10} + \frac{1}{665280}x^{11} + \frac{1}{1995840}x^{12} + \frac{1}{2520}x^7 + \frac{1}{5040}x^9 - \frac{1}{1680}x^8$$

$$y_{22}(x) = -\frac{1}{415800}x^{11} + \frac{1}{1995840}x^{12} - \frac{1}{181440}x^{10} + \frac{1}{630}x^7 - \frac{1}{1120}x^8 - \frac{1}{7560}x^9$$

$$y_{32}(x) = \frac{1}{181440}x^{10} - \frac{1}{1663200}x^{12} + \frac{1}{665280}x^{11} + \frac{1}{3360}x^8 - \frac{1}{15120}x^9 + \frac{1}{840}x^7$$

$$y_{13}(x) = -\frac{1}{165110400}x^{14} - \frac{1}{3027024000}x^{15} + \frac{1}{311351040}x^{13} + \frac{1}{75600}x^{10} - \frac{1}{1995840}x^{12}$$

$$y_{23}(x) = \frac{1}{35380800}x^{13} - \frac{17}{136218000}x^{15} - \frac{1}{242161920}x^{14} + \frac{1}{415800}x^{11} - \frac{1}{1995840}x^{12}$$

$$y_{33}(x) = -\frac{1}{172972800}x^{13} + \frac{17}{3632428800}x^{14} - \frac{1}{5448643200}x^{15} - \frac{1}{181440}x^{10} + \frac{1}{1663200}x^{12}$$

and so on, in the same manner, the rest components can be obtained.

The sum of the first four terms is:

$$y_1(x) = x - \frac{1}{165110400}x^{14} - \frac{1}{3027024000}x^{15} + \frac{1}{311351040}x^{13}$$

$$y_2(x) = x^2 + \frac{1}{35380800}x^{13} - \frac{17}{136216080000}x^{15} - \frac{1}{242161920}x^{14}$$

$$y_3(x) = x^3 - \frac{1}{172972800}x^{13} + \frac{17}{136216080000}x^{14} - \frac{1}{5448643200}x^{15}.$$

.

Example 3

Consider the system of nonlinear third-order Volterra integro-differential equation:

$$y_1'''(x) = -2x - 2x^3 - \frac{2}{5}x^5 + \int_0^x \left(y_1^2(t) + y_2^2(t)\right)dt, \quad (16)$$

$$y_2'''(x) = -\frac{2}{3}x^3 - -\frac{1}{5}x^5 + \int_0^x (x-t)\left(y_1^2(t) - y_2^2(t)\right)dt,$$

with the initial conditions:

$y_1(0) = 1,\ y_1'(0) = 1, y_1''(0) = 2$

$y_2(0) = 1,\ -y_2'(0) = 1,\ y_2''(0) = 2$

As the above examples, from (3.16), we obtain:

The system of integro-differential equation is equivalent to the system of integral equation:

$$y_1(x) = 1 + x + x^2 - \frac{1}{12}x^4 - \frac{1}{60}x^6 - \frac{1}{840}x^8 + I_x^3\left[\int_0^x \left(y_1{}^2(t) + y_2{}^2(t)\right)dt\right]$$

$$y_2(x) = 1 - x + x^2 - \frac{1}{180}x^6 - \frac{1}{1680}x^8$$

$$+I_x^3\left[\int_0^x (x-t)\left(y_1{}^2(t) - y_2{}^2(t)\right)dt\right]$$

Let

$$N_1(y) = I_x^3\left[\int_0^x \left(y_1{}^2(t) + y_2{}^2(t)\right)dt\right],$$

$$N_2(y) = I_x^3\left[\int_0^x (x-t)\left(y_1{}^2(t) - y_2{}^2(t)\right)dt\right]$$

$$y_{10}(x) = 1 + x + x^2 - \frac{1}{12}x^4 - \frac{1}{60}x^6 - \frac{1}{840}x^8$$

$$y_{20}(x) = 1 - x + x^2 - \frac{1}{180}x^6 - \frac{1}{1680}x^8$$

$$y_{11}(x) = I_x^3\left[\int_0^x \left(y_{10}{}^2(t) + y_{20}{}^2(t)\right)dt\right]$$

$$= \frac{1}{12}x^4 + \frac{1}{6567734400}x^{20} + \frac{1}{1586304000}x^{18} + \frac{23}{1981324800}x^{16} - \cdots$$

$$y_{21}(x) = I_x^3\left[\int_0^x (x-t)\left(y_{10}{}^2(t) - y_{20}{}^2(t)\right)dt\right]$$

$$= \frac{1}{2297320704000}x^{21} + \frac{1}{42195686400}x^{10} + \frac{1}{16841260800}x^8 + \cdots$$

$$y_{12}(x) = \frac{1}{10080}x^8 + \frac{31601}{159659194286592000}x^{22} - \frac{1}{152527239365702506905600000}x^{39} - \cdots$$

$$y_{22}(x) = \frac{134179}{17562511371525120000}x^{22}$$

$$+ \frac{112417}{7164564811401530886145228000000}x^{42} + \cdots$$

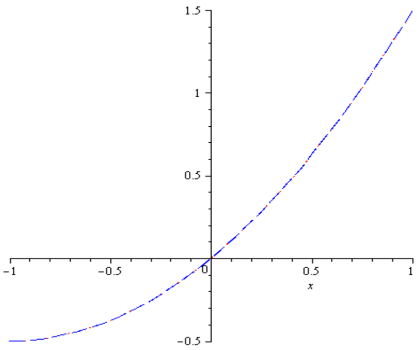
$$\cdots$$

and so on, in the same manner, the rest components can be obtained.

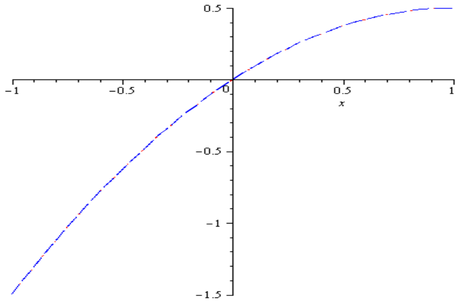
The sum of the first three terms is:

$$y_1(x) = 1 + x + x^2 - \frac{31601}{15965919428659200}x^{22} + \cdots$$

$$y_2(x) = 1 - x + x^2 + \frac{134179}{175625113715251200000}x^{22} + \cdots$$

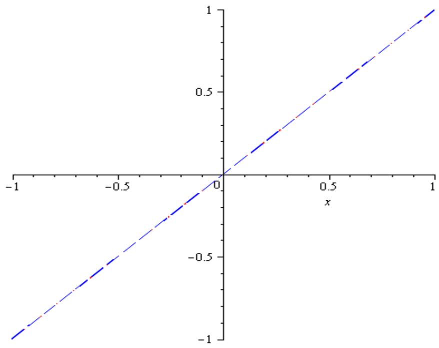


(a)

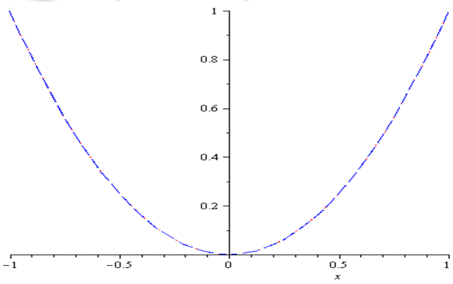


(b)

Figure 1: Exact and approximate solutions for (a) $y_1(x)$ and (b) $y_2(x)$ of Eq. (14), where the red and the blue represent the approximate and exact solutions respectively.



(a)



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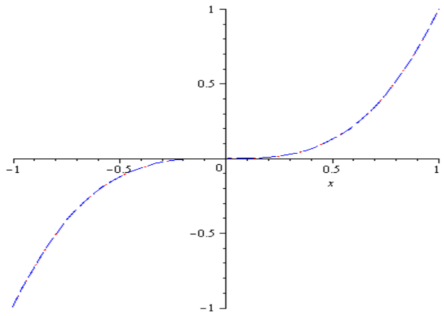

(c)

Figure 2: Exact and approximate solution for (a) $y_1(x)$, (b) $y_2(x)$ and (c) $y_3(x)$ of Eq. (4.5), where red and blue represent the approximate and exact solutions respectively.

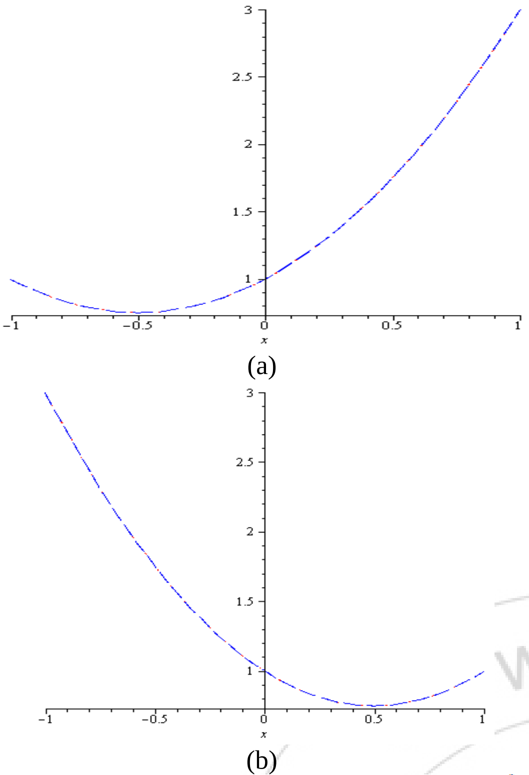

Figure 3: Exact and approximate solutions for (a) $y_1(x)$ and (b) $y_2(x)$ for Eq. (16), where the red and blue represent the approximate and exact solution respectively.

5. Conclusion

In this paper, we successfully applied the new iterative method to find the solution of system of the nth-order linear and nonlinear Volterra integro-differential equations. The present method converts a system of Volterra integro-differential equation to a system of Volterra integral equation. It is clear from the graphs that the solutions agree well with the exact solutions for these equations. The results showed that the method is very accurate and simple.

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