International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2013): 4.438

On a Subclass of Spiral-Like Functions by Applying Fractional Calculus

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Abstract: ***In this paper, we study an application of the fractional calculus techniques for the subclass of Spiral-Like functions $\mathcal{R}_\delta(\beta,\alpha,\gamma)$. Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get some geometric properties, like, extreme points, radii of starlikeness, convexity and close-to-convexity, closure theorems and partial sum.***

Mathematics Subject Classification: 30C45

Keywords: Convex function, Spiral-Like function, Fractional calculus, Distortion Theorem, Radius of Starlikeness, Partial Sums.

1. Introduction

Let $\mathcal{R}$ denote the class of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n,\quad (a_n\geq 0, n\in N=\{1,2,3,\ldots\}),\quad (1)$$

which are analytic and univalent in the unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. For β real, $|\beta|<\frac{\pi}{2}$, a function f in the form (1) is said to be in $\mathcal{R}(\beta)$, the class of $\beta-$ spiral $-$ Like function if

$$Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right)>0, z\in U.\quad (2)$$

For $\beta=0,\mathcal{R}(0)\equiv\mathcal{R}$ is the well-known class of functions starlike with respect to the origin, for $\beta\neq 0$, it is know that $\mathcal{R}(\beta)$ is not contained in $\mathcal{R}$. In fact the class $\mathcal{R}(\beta)$ was introduced and shown to be a subfamily of $\mathcal{R}$ by Spačěk [7]. Later, Zomorski [9] obtained sharp coefficient bounds for the class. Recently Several authors studied Spiral-Like function for different classes, like, Atshan [1].

Definition (1)[8]: The fractional integral of order δ $(\delta>0)$, is defined by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(t)}{(z-t)^{1-\delta}}dt,\quad (3)$$

where f is an analytic function in a simply $-$ connected region of the z $-$plane containing the origin, and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real, when $Re(z-t)>0$.

Definition (2)[8]: The fractional derivative of order $\delta(0\leq\delta<1$ is defined by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\int_0^z\frac{f(t)}{(z-t)^{\delta}}dt,\quad (4)$$

where f(z) is as in Definition (1) and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition (1).

Definition (3) [8]: [Under the condition of Definition 2] the fractional derivative of order $n+\delta,(n=0,1,2,\ldots)$ is defined by

$$D_z^{n+\delta}f(z)=\frac{d^n}{dz^n}D_z^{\delta}f(z).$$

From Definition (1) and Definition (2) by applying a simple calculation, we get

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(2+\delta)}z^{\delta+1}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n+\delta},\quad (5)$$

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(2-\delta)}z^{\delta-1}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n+\delta},\quad (6)$$

Definition (4): We introduce a new subclass of $\mathcal{R}(\beta)$ as functions in the form (1) that satisfy the inequality:

$$\left|\frac{\frac{zf''(z)}{f'(z)}}{2\alpha\left[\frac{zf''(z)}{f'(z)}+(1-\gamma)e^{-i\beta}\cos\beta\right]+\frac{zf''(z)}{f'(z)}}\right|<1,\qquad \text{for } z\in U,$$

where $0\leq\gamma<1, 0<\gamma\leq 1, |\beta|<\frac{\pi}{2}$. We denote for our class by $\mathcal{R}_\delta(\beta,\alpha,\gamma)$.

2. Main Results

In the following theorem, we obtain the coefficient inequality for the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$.

Theorem (1): Let $f(z)\in\mathcal{R}(\beta)$. Then $f(z)$ is in the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]a_n\leq\alpha(1-\gamma)|e^{i\beta}\cos\beta|,\quad (7)$$

where $0\leq\gamma<1, 0<\gamma\leq 1, |\beta|<\frac{\pi}{2}$.

The result (7) is sharp for the function $f(z)$ given by:

$$f(z)=z-\frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}z^n.\quad (8)$$

Proof: Let (7) holds true and$|z|=1$, we have

$$|zf''(z)|-|2\alpha[zf''(z)+(1-\gamma)e^{-i\beta}\cos\beta f'(z)]+zf''(z)|$$

$$=\left|-\sum_{n=2}^{\infty}n(n-1)a_n z^n\right|-\left|2\alpha(1-\gamma)e^{-i\beta}\cos\beta z-\sum_{n=2}^{\infty}n[(n-1)(1+2\alpha)+2\alpha(1-\gamma)e^{-i\beta}\cos\beta]a_n z^n\right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1)a_n - 2\alpha(1-\gamma)|e^{-i\beta}cos\beta| + \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta}cos\beta]a_n$$

$$= \sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta}cos\beta|]a_n - \alpha(1-\gamma)|e^{i\beta}cos\beta| \leq 0,$$

by hypothesis. Thus by Maximum modules theorem $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$.
Conversely, assume that

$$\left|\frac{\frac{zf''(z)}{f'(z)}}{2\alpha\left[\frac{zf''(z)}{f'(z)} + (1-\gamma)e^{-i\beta}cos\beta f'(z)\right] + \frac{zf''(z)}{f'(z)}}\right| = \left|\frac{zf''(z)}{2\alpha[zf''(z) + (1-\gamma)e^{-i\beta}cos\beta f'(z)] + zf''(z)}\right|$$

$$= \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{2\alpha(1-\gamma)e^{-i\beta}cos\beta - \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta}cos\beta]a_n z^{n-1}}\right| < 1.$$

Since $Re(z) \leq |z|$ for all z, we have

$$Re\left(\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{2\alpha(1-\gamma)e^{-i\beta}cos\beta - \sum_{n=2}^{\infty} n[(n-1)(1+2\alpha) + 2\alpha(1-\gamma)e^{-i\beta}cos\beta]a_n z^{n-1}}\right) < 1, \quad (9)$$

we can choose value of z on the real axis so that $f(z)$ is real. Let $z \to 1^-$, through real values, so we write (9) as

$$\sum_{n=2}^{\infty} n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta}cos\beta|]a_n \leq \alpha(1-\gamma)|e^{i\beta}cos\beta|. \blacksquare$$

Corollary (1): Let $f(z) \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$. Then

$$a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{n[(n-1)(1+\alpha) + \alpha(1-\gamma)|e^{i\beta}cos\beta|]}, \quad n \geq 2.$$

Theorem (2): Let $f(z) \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$. Then

$$|D_z^{-\delta}f(z)| \leq \frac{1}{\Gamma(2+\delta)}|z|^{\delta+1}\left[1 + \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{(2+\delta)[1+\alpha(1-\gamma)|e^{i\beta}cos\beta|]}|z|\right], \quad (10)$$

and

$$|D_z^{-\delta}f(z)| \geq \frac{1}{\Gamma(2+\delta)}|z|^{\delta+1}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{(2+\delta)[1+\alpha(1-\gamma)|e^{i\beta}cos\beta|]}|z|\right]. \quad (11)$$

The inequalities in (10) and (11) are attained for the function $f(z)$ given by:

$$f(z) = z - \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{2[1+\alpha(1+(1-\gamma))|e^{i\beta}cos\beta|]}z^2. \quad (12)$$

Proof: by using Theorem (1), we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{2[1+\alpha(1+(1-\gamma))|e^{i\beta}cos\beta|]}, \quad (13)$$

by Definition (3), we have

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(2+\delta)}z^{1+\delta} - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1+\delta)}a_n z^{n+\delta},$$

and

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z) = z - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}a_n z^n$$

$$= z - \sum_{n=2}^{\infty}\theta(n)\,a_n z^n. \quad (14)$$

where

$$\theta(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)}.$$

We know that $\theta(n)$ is a decreasing function of n and

$$0 < \theta(n) < \theta(2) = \frac{2}{2+\delta}.$$

Using (13) and (14), we have

$$|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| \leq |z| + \theta(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leq |z| + \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{(2+\delta)[1+\alpha(1+(1-\gamma))|e^{i\beta}cos\beta|]}|z|^2,$$

which gives (10), we also have

$$|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta}f(z)| \geq |z| - \theta(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geq |z| - \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{(2+\delta)[1+\alpha(1+(1-\gamma))|e^{i\beta}cos\beta|]}|z|^2,$$

which gives (11).■

Theorem(3): Let $f(z) \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$. Then

$$|D_z^{\delta}f(z)| \leq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 + \frac{\alpha(1-\gamma)|e^{i\beta}cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta}cos\beta|]}|z|\right], \quad (15)$$

and

$$|D_z^\delta f(z)| \geq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right]. \quad (16)$$

The inequalities in (15) and (16) are attained for the function $f(z)$ given by (12).

Proof: From Definition (3), we have

$$D_z^\delta f(z) = \frac{1}{\Gamma(2-\delta)}z^{\delta+1} - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n-\delta}$$

and

$$\Gamma(2-\delta)z^\delta D_z^\delta f(z) = z - \sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}a_n z^n = z - \sum_{n=2}^{\infty}\theta(n)\, a_n z^n$$

where $\Phi(n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$ for $n \geq 2$, $\Phi(n)$ is a decreasing of n, then

$$\Phi(n) \leq \Phi(2) = \frac{2}{2-\delta}.$$

Also by using Theorem (1), we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{2[1+\alpha(1+(1-\gamma))|e^{i\beta}\cos\beta|]},$$

thus

$$|\Gamma(2-\delta)z^\delta D_z^\delta f(z)| \leq |z| - \theta(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leq |z| - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2-\delta)[1+\alpha(1+(1-\gamma))|e^{i\beta}\cos\beta|]}|z|^2.$$

Then

$$|D_z^\delta f(z)| \leq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right],$$

and by the same way, we obtain

$$|D_z^\delta f(z)| \geq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right]. ■$$

Corollary (2): For every $f(z) \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$, we have

$$\frac{|z|^2}{2}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{3[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right] \leq \left|\int_0^z f(t)\,dt\right| \leq \frac{|z|^2}{2}\left[1 + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{3[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right], (17)$$

and

$$|z|\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{2[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right] \leq |f(z)| \leq |z|\left[1 + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{2[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}|z|\right] \quad (18)$$

Proof: i) By Definition (1) and Theorem (2) for $\delta = 1$, we have

$$D_z^{-1}f(z) = \int_0^z f(t)\,dt,$$

the result is true.

ii) By Definition (2) and Theorem (2) for $\delta = 0$, we have

$$D_z^0 f(z) = \frac{d}{dz}\int_0^z f(t)\,dt = f(z),$$

the result is true.■

Corollary (3): $D_z^{-\delta}f(z)$ and $D_z^\delta f(z)$ are included in the disk with center at origin and radii

$$\frac{1}{\Gamma(2+\delta)}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2+\delta)[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\right], \frac{1}{\Gamma(2-\delta)}\left[1 - \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{(2-\delta)[1+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\right].$$

In the following theorem, we obtain the extreme points of the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$.

Theorem (4): Let $f_1(z) = z, f_n(z) = z + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}z^n, (n \geq 2, (19)$ *where* $n \in N$, $0 \leq \gamma < 1$, $0 < \gamma \leq 1$, $\beta < \pi 2$. Then the function $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$ if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty}\mu_n f_n(z), \quad (20)$$

where $\mu_n \geq 0$, $\sum_{n=1}^{\infty}\mu_n = 1$ *or* $1 = \mu_1 + \sum_{n=2}^{\infty}\mu_n$.

Proof: Let $f(z)$ can be expressed as in (20). Then

$$f(z) = \sum_{n=1}^{\infty}\mu_n f_n(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty}\mu_n f_n(z)$$

$$= \mu_1 z + \sum_{n=2}^{\infty}\mu_n\left(z + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\right)$$

$$= z\left(\mu_1 + \sum_{n=2}^{\infty}\mu_n\right) + \sum_{n=2}^{\infty}\frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\mu_n z^n$$

$$= z + \sum_{n=2}^{\infty}\frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\mu_n z^n$$

$$= z - \sum_{n=2}^{\infty}h_n z^n,$$

where

$$h_n = \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\mu_n.$$

Thus

$$\sum_{n=2}^{\infty}h_n\frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|} = \sum_{n=2}^{\infty}\frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}\mu_n \times \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}$$

$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 < 1.$$

Therefore, we have $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$.

Conversely, suppose that $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$. Then by (7), we have

$$a_n \le \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}, \quad (n \ge 2)$$

we may set,

$$\mu_n = \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}, \quad (n \ge 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

Then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$= z + \sum_{n=2}^{\infty} \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]} \mu_n z^n$$

$$= z + \sum_{n=2}^{\infty} \mu_n (z - f_n(z)) = \left(1 - \sum_{n=2}^{\infty} \mu_n\right) z + \sum_{n=2}^{\infty} \mu_n f_n(z)$$

$$= z\mu_1 + \sum_{n=2}^{\infty} \mu_n f_n(z) = \sum_{n=1}^{\infty} \mu_n f_n(z). \blacksquare$$

In the following theorems, we obtain radii of starlikeness, convexity and close-to-convexity of the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$.

Theorem (5): If $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$, then f is starlike of order δ $(0 \le \delta < 1)$ in the disk $|z| < r_1(\beta,\alpha,\gamma,\delta)$, where

$$r_1(\beta,\alpha,\gamma,\delta) = \inf_n \left\{ \frac{(1-\delta)n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right\}^{\frac{1}{n-1}}, n \ge 2. \quad (21)$$

The result is sharp with extremal function f given by

$$f(z) = z + \frac{\alpha(1-\gamma)|e^{i\beta}\cos\beta|}{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]} z^n. \quad (22)$$

Proof: It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1 - \delta, \quad (0 \le \delta < 1)$$

for $|z| < r_1(\beta,\alpha,\gamma,\delta)$. We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zf'(z) - f(z)}{f(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} a_n (n-1) z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \le \frac{\sum_{n=2}^{\infty} a_n (n-1)|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{(n-\delta)a_n |z|^{n-1}}{(1-\delta)} \le 1. \quad (23)$$

Hence, by Theorem (1), (23) will be true if

$$\frac{(n-\delta)}{(1-\delta)} |z|^{n-1} \le \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|},$$

or equivalently

$$|z| \le \left(\frac{n(1-\delta)[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_1(\beta,\alpha,\gamma,\delta)$, we get the desired result. ■

Theorem (6): If $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$, then f is convex of order δ $(0 \le \delta < 1)$ in the disk $|z| < r_2(\beta,\alpha,\gamma,\delta)$, where

$$r_2(\beta,\alpha,\gamma,\delta) = \inf_n \left\{ \frac{(1-\delta)n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{n(n-\delta)\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right\}^{\frac{1}{n-1}}, n \ge 2. \quad (24)$$

The result is sharp with extremal function f given by (22).

Proof: It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} \right| \le 1 - \delta, \quad (0 \le \delta < 1)$$

for $|z| < r_2(\beta,\alpha,\gamma,\delta)$. We have

$$\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1}}.$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)a_n |z|^{n-1}}{(1-\delta)} \le 1. \quad (25)$$

Hence, by Theorem (1), (25) will be true if

$$\frac{n(n-\delta)}{(1-\delta)} |z|^{n-1} \le \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|},$$

or equivalently,

$$|z| \le \left(\frac{(1-\delta)n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{n(n-\delta)\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_2(\beta,\alpha,\gamma,\delta)$, we get the desired result. ■

Theorem (7): If $f \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$, then f is close-to-convex of order δ $(0 \le \delta < 1)$ in the disk $|z| < r_3(\beta,\alpha,\gamma,\delta)$, where

$$r_3(\beta,\alpha,\gamma,\delta) = \inf_n \left\{ \frac{(1-\delta)[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right\}^{\frac{1}{n-1}}, n \ge 2. \quad (26)$$

The result is sharp with extremal function f given by (22)

Proof: It is sufficient to show that

$$|f'(z) - 1| \le 1 - \delta, \quad (0 \le \delta < 1)$$

for $|z| < r_3(\beta,\alpha,\gamma,\delta)$. We have

$$|f'(z) - 1| = \left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \le \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression above is bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} \frac{n a_n |z|^{n-1}}{(1-\delta)} \le 1. \quad (27)$$

Hence, by Theorem (1), (27) will be true if

$$\frac{n}{(1-\delta)} |z|^{n-1} \le \frac{n[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|},$$

or equivalently,

$$|z| \le \left(\frac{(1-\delta)[(n-1)(1+\alpha)+\alpha(1-\gamma)|e^{i\beta}\cos\beta|]}{\alpha(1-\gamma)|e^{i\beta}\cos\beta|} \right)^{\frac{1}{n-1}}.$$

Setting $|z| = r_3(\beta,\alpha,\gamma,\delta)$, we get the desired result. ■

In the following Theorem, we prove the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$ is closed under linear combination.

Theorem (8): Let the function $f_j(z) \in \mathcal{R}_\delta(\beta,\alpha,\gamma)$ defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n, \qquad \left(a_{n,j} \geq 0, j = 1,2,3,\dots,l\right).$$

Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{l} c_j\, f_j(z),$$

is in the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$, where $\sum_{j=1}^{l} c_j = 1,\ c_j \geq 0$.

Proof: By definition of $h(z)$, we have

$$h(z) = \left[\sum_{j=1}^{l} c_j\right] z - \sum_{n=2}^{\infty}\left[\sum_{j=1}^{l} c_j\, a_{n,j}\right] z^n. \tag{28}$$

Further, since $f_j(z)$ are in the class $\mathcal{R}_\delta(\beta,\alpha,\gamma)$ for every $j = 1,2,3,\dots,l$.

Hence, we can see that

$$\sum_{n=2}^{\infty}\left(n\left[(n-1)(1+\alpha)+\alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|\right]\right)\left[\sum_{j=1}^{l} c_j\, a_{n,j}\right]$$
$$= \sum_{j=1}^{l} c_j\left[\sum_{n=2}^{\infty} n\left[(n-1)(1+\alpha)+\alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|\right] a_{n,j}\right]$$
$$\leq \alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|\sum_{j=1}^{l} c_j = \alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|. \blacksquare$$

Let $f \in \mathcal{R}$ be a function of the form (1). Motivated by Silverman [3] and Silvia [5], see also [4], [6], we define the partial sums f_m defined by

$$f_m(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (m \in N). \tag{29}$$

Theorem (9): Let $f \in \mathcal{R}$ be given by (1) and define the partial sums $f_1(z)$ and $f_m(z)$ as follows: $f_1(z) = z$ and

$$f_m(z) = z - \sum_{n=2}^{m} a_n\, z^n, \qquad (m > 2). \tag{30}$$

Also suppose that

$$\sum_{n=2}^{\infty} d_n a_n \leq 1, \quad \left(d_n = \frac{n\left[(n-1)(1+\alpha)+\alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|\right]}{\alpha(1-\gamma)\left|e^{i\beta}\cos\beta\right|}\right). \tag{31}$$

Then, we have

$$Re\left\{\frac{f(z)}{d_m(z)}\right\} > 1 - \frac{1}{d_m}, \tag{32}$$

and

$$Re\left\{\frac{f_m(z)}{f(z)}\right\} > \frac{d_{m+1}}{1+d_{m+1}}\ . \tag{33}$$

Each of the bounds in (32) and (33) is the best possible for $m \in N$.

Proof: For the coefficients d_n given by (31), it is not difficult to verify that

$d_{n+1} > d_n > 1$, $n = 2,3 \dots$. Therefore, we have

$$\sum_{n=2}^{m} a_n + d_m \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1. \tag{34}$$

By setting

$$g_1(z) = d_m\left[\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_m}\right)\right] = 1 + \frac{d_m \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^{m} a_n z^{n-1}}, \tag{35}$$

and applying (34), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_m \sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{m} a_n - d_m \sum_{n=m+1}^{\infty} a_n} \leq 1, \quad (z \in U)$$

which readily yields the left assertion (32). If we take

$$f(z) = z - \frac{z^m}{d_m}, \tag{36}$$

then

$$\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{d_m} \longrightarrow 1 - \frac{1}{d_m} (z \longrightarrow 1^-),$$

Similarly, if we take

$$g_2(z) = (1+d_m)\left[\frac{f_m(z)}{f(z)} - \frac{d_m}{1+d_m}\right]$$

and making use of (34), we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum_{n=m+1}^{\infty} a_n}{2 - 2\sum_{n=2}^{m} a_n + (1-d_m)\sum_{n=m+1}^{\infty} a_n} \leq 1, \tag{37}$$

which leads us to the assertion (33). The bound in (33) is sharp for each $m \in \mathbb{N}$ with the function given by (36).

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