On a Subclass of Spiral-Like Functions by Applying Fractional Calculus

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Abstract: In this paper, we study an application of the fractional calculus techniques for the subclass of Spiral-Like functions \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \). Distortion theorems for the fractional derivative and fractional integration are obtained. Also we get some geometric properties, like, extreme points, radii of starlikeness, convexity and close-to-convexity, closure theorems and partial sum.

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1. Introduction

Let \( \mathcal{R} \) denote the class of functions of the form:

\[
f(z) = z - \sum_{n=0}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in N = \{1,2,3,...\}), \quad (1)
\]

which are analytic and univalent in the unit disk \( U = \{z \in \mathbb{C}: |z| < 1\} \). For \( \beta \) real, \( |\beta| < \frac{\pi}{2} \), a function \( f \) in the form (1) is said to be in \( \mathcal{R}(\beta) \), the class of \( \beta \) - spiral – Like function if

\[
Re \left( e^{i\beta} \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U. \quad (2)
\]

For \( \beta = 0, \mathcal{R}(0) \equiv \mathcal{R} \) is the well-known class of functions starlike with respect to the origin, for \( \beta \neq 0 \), it is known that \( \mathcal{R}(\beta) \) is not contained in \( \mathcal{R} \). In fact the class \( \mathcal{R}(\beta) \) was introduced and shown to be a subfamily of \( \mathcal{R} \) by Spaček [7]. Later, Zomorouk [9] obtained sharp coefficient bounds for the class. Recently Several authors studied Spiral-Like function for different classes, like, Atshan [1].

Definition (1)[8]: The fractional integral of order \( \delta (\delta > 0) \), is defined by

\[
D_2^{-\delta}f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt, \quad (3)
\]

where \( f \) is an analytic function in a simply – connected region of the \( z \)-plane containing the origin, and the multiplicity of \( (z-t)^{-1} \) is removed by requiring \( \log(z-t) \) to be real, when \( \text{Re}(z-t) > 0 \).

Definition (2)[8]: The fractional derivative of order \( \delta (0 \leq \delta < 1) \) is defined by

\[
D_2^{\delta}f(z) = \frac{1}{\Gamma(1-\delta)} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt, \quad (4)
\]

where \( f(z) \) is as in Definition (1) and the multiplicity of \( (z-t)^{-\delta} \) is removed like Definition (1).

Definition (3) [8]: [Under the condition of Definition 2] the fractional derivative of order \( n + \delta, (n = 0,1,2,...) \) is defined by

\[
D_2^{n+\delta}f(z) = \frac{d^n}{dz^n} D_2^{\delta}f(z).
\]

From Definition (1) and Definition (2) by applying a simple calculation, we get

\[
D_2^{\delta}f(z) = \frac{1}{\Gamma(2-\delta)} z^{\delta+1} - \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n+\delta}, \quad (5)
\]

\[
D_2^{-\delta}f(z) = \frac{1}{\Gamma(2-\delta)} z^{\delta-1} - \sum_{n=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n+\delta}, \quad (6)
\]

Definition (4): We introduce a new subclass of \( \mathcal{R}(\beta) \) as functions in the form (1) that satisfy the inequality:

\[
\left| 2\alpha \left[ \frac{z f'(z)}{f(z)} + (1-\gamma) e^{-i\beta} \cos \beta \right] + \frac{zf'(z)}{f(z)} \right| < 1, \quad \text{for } z \in U,
\]

where \( 0 \leq \gamma < 1, 0 < \gamma \leq 1, |\beta| < \frac{\pi}{2} \). We denote for our class by \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \).

2. Main Results

In the following theorem, we obtain the coefficient inequality for the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \).

Theorem (1): Let \( f(z) \in \mathcal{R}(\beta) \). Then \( f(z) \) is in the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \) if and only if

\[
\sum_{n=0}^{\infty} n \left| (n+1) (1+\alpha) + \alpha (1-\gamma) e^{i\theta} \cos \beta \right| a_n \leq \alpha (1-\gamma) e^{i\theta} \cos \beta \leq \alpha (1+\alpha) \left| e^{i\theta} \cos \beta \right| z^n.
\]

where \( 0 \leq \gamma < 1, 0 < \gamma \leq 1, |\beta| < \frac{\pi}{2} \).

The result (7) is sharp for the function \( f(z) \) given by:

\[
f(z) = z - \alpha (1-\gamma) e^{i\theta} \cos \beta \left[ (n-1) (1+\alpha) + \alpha (1-\gamma) e^{i\theta} \cos \beta \right] z^n.
\]

Proof: Let (7) holds true and \( |z| = 1 \), we have

\[
|zf'(z)| - |2\alpha z f'(z) + (1-\gamma) e^{-i\beta} \cos \beta f'(z) + zf'(z)|
\]


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Conversely, assume that
\[ 1 - \alpha \geq z \Gamma (\alpha) \Gamma (\beta) + 1 + \alpha (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1) \] 
by hypothesis. Thus by Maximum modulus theorem \( f \in \mathcal{R}_a (\beta, \alpha, \gamma) \).

Corollary (1): Let \( f (z) \in \mathcal{R}_a (\beta, \alpha, \gamma) \). Then
\[ a_n \leq n [(n - 1)(1 + \alpha) + \alpha (1 - \gamma) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1)] \]

Theorem (2): Let \( f (z) \in \mathcal{R}_a (\beta, \alpha, \gamma) \). Then
\[ D_z^\delta f(z) = \frac{1}{\Gamma (2 + \delta)} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

We know that \( \theta (n) \) is a decreasing function of \( n \) and
\[ 0 < \theta (n) < \theta (2) = \frac{2}{2 + \delta} \]

Using (13) and (14), we have
\[ |\Gamma (2 + \delta) z^{-\delta} D_z^{-\delta} f(z) | \leq |z| + |\theta (2)| |z|^{2} \sum_{n=2}^{\infty} a_n \]

The inequalities in (10) and (11) are attained for the function \( f(z) \) given by:
\[ f(z) = z^2 \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

Proof: by using Theorem (1), we have
\[ D_z^{-\delta} f(z) \leq \frac{1}{\Gamma (2 + \delta)} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta}, \]

and
\[ \sum_{n=2}^{\infty} a_n \leq a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1) \]

by Definition (3), we have
\[ \frac{a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1)}{2 + \delta} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

and
\[ \sum_{n=2}^{\infty} a_n \leq a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1) \]

by Definition (3), we have
\[ \frac{a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1)}{2 + \delta} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

Theorem (3): Let \( f(z) \in \mathcal{R}_a (\beta, \alpha, \gamma) \). Then
\[ |D_z^\delta f(z) | \leq \frac{1}{\Gamma (2 + \delta)} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

The inequalities in (10) and (11) are attained for the function \( f(z) \) given by:
\[ f(z) = z^2 \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

Proof: by using Theorem (1), we have
\[ D_z^{-\delta} f(z) \leq \frac{1}{\Gamma (2 + \delta)} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta}, \]

and
\[ \sum_{n=2}^{\infty} a_n \leq a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1) \]

by Definition (3), we have
\[ \frac{a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1)}{2 + \delta} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]

and
\[ \sum_{n=2}^{\infty} a_n \leq a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1) \]

by Definition (3), we have
\[ \frac{a (1 - \gamma) \Gamma (\alpha) \Gamma (\beta) \Gamma (\alpha + \beta) \Gamma (\alpha + \beta + 1)}{2 + \delta} \frac{\Gamma (n + 1) + \Gamma (1 + \delta)}{\Gamma (n + 1 + \delta)} a_n z^{n+\delta} \]
Proof: From Definition (3), we have
\[
D^\theta f(z) = \frac{1}{\Gamma(2-\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta}
\]
and
\[
\Gamma(2-\delta) z^\delta D^\theta f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n
\]
where \(\Phi(n) = \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\) for \(n \geq 2\), \(\Phi(n)\) is a decreasing function of \(n\), then
\[
\Phi(n) \leq 2 - \delta
\]
Also by using Theorem (1), we have
\[
\sum_{n=2}^{\infty} a_n \leq \frac{a(1-\gamma) e^{i \beta \cos \theta}}{2(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}}
\]
Thus
\[
\left|\Gamma(2-\delta) z^\delta D^\theta f(z)\right| \leq |z| - \theta(2)|z|^2 \sum_{n=2}^{\infty} a_n
\]
\[
\leq |z| - \frac{a(1-\gamma) e^{i \beta \cos \theta}}{(2-\delta)(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|^2.
\]
Then
\[
\left|D^\theta f(z)\right| \leq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 - \frac{a(1-\gamma) e^{i \beta \cos \theta}}{(2-\delta)(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|\right].
\]
and by the same way, we obtain
\[
\left|D^\theta f(z)\right| \geq \frac{1}{\Gamma(2-\delta)} |z|^{1-\delta} \left[1 - \frac{a(1-\gamma) e^{i \beta \cos \theta}}{(2-\delta)(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|\right].
\]
\[
\text{Corollary (2): For every } f(z) \in R(z(\beta, \alpha, \gamma)), \text{ we have }
\]
\[
\left|f(z)\right|^2 \left[1 - \frac{a(1-\gamma) e^{i \beta \cos \theta}}{3(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|\right] \leq \left|\int_0^z f(t) dt\right| \leq \frac{1}{2} \left| \int_0^z f(t) dt\right| \frac{1}{2} + \frac{a(1-\gamma) e^{i \beta \cos \theta}}{3(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|, (17)
\]
and
\[
\left|z\right| \left[1 - \frac{a(1-\gamma) e^{i \beta \cos \theta}}{2(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|\right] \leq \left|f(z)\right| \leq \left|z\right| \left[1 + \frac{a(1-\gamma) e^{i \beta \cos \theta}}{2(1+a(1-(1-\gamma))) e^{i \beta \cos \theta}} |z|\right]. (18)
\]
Proof: i) By Definition (1) and Theorem (2) for \(\delta = 1\), we have
\[
D_1^{-1} f(z) = \int_0^z f(t) dt,
\]
the result is true.
ii) By Definition (2) and Theorem (2) for \(\delta = 0\), we have
\[
D_0^\theta f(z) = \frac{d}{dz}\int_0^z f(t) dt = f(z),
\]
the result is true.
\[
\text{Corollary (3): } D_1^{-\delta} f(z) \text{ and } D_0^\theta f(z) \text{ are included in the disk with center at origin and radii }
\]
\[
\text{Proof: Let } f(z) \text{ be expressed as in (20). Then }
\]
\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),
\]
where \(\mu_n \geq 0\), \(\sum_{n=1}^{\infty} \mu_n = 1\) or \(1 = \mu_1 + \sum_{n=2}^{\infty} \mu_n\).
\[
\text{Proof: Let } f(z) \text{ can be expressed as in (20). Then }
\]
\[
f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z)
\]
where \(\mu_1 \geq 0\), \(\sum_{n=1}^{\infty} \mu_n = 1\) or \(1 = \mu_1 + \sum_{n=2}^{\infty} \mu_n\).
Therefore, we have \( f \in \mathcal{R}_\delta(\beta, \alpha, \gamma) \).

Conversely, suppose that \( f \in \mathcal{R}_\delta(\beta, \alpha, \gamma) \). Then by (7), we have
\[
a_n \leq \frac{a(1-\gamma)|e^{i\beta}cos\beta|}{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]} \quad (n \geq 2)
\]
we may set,
\[
\mu_n = \frac{a(1-\gamma)|e^{i\beta}cos\beta|}{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]} , \quad (n \geq 2)
\]
and
\[
\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.
\]

Then
\[
f(z) = z + \sum_{n=2}^{\infty} \frac{a_n z^n}{(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} \mu_n z^n
\]
\[
= z + \sum_{n=2}^{\infty} \mu_n (f_n(z)) = \left( 1 - \sum_{n=2}^{\infty} \mu_n \right) z + \sum_{n=2}^{\infty} \mu_n f_n(z)
\]
\[
= z\mu_1 + \sum_{n=2}^{\infty} \mu_n f_n(z) = \sum_{n=2}^{\infty} \mu_n f_n(z).
\]

In the following theorems, we obtain radii of starlikeness, convexity and close-to-convexity of the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \).

Theorem (5): If \( f \in \mathcal{R}_\delta(\beta, \alpha, \gamma) \), then \( f \) is starlike of order \( \delta \) \( (0 \leq \delta < 1) \) in the disk \( |z| < r_1(\beta, \alpha, \gamma, \delta) \), where
\[
r_1(\beta, \alpha, \gamma, \delta) = \text{inf}\left\{ \frac{(1-\delta)n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta}cos\beta|} \right\}^{\frac{1}{n-1}}
\]
\[
\geq 2. \quad (21)
\]

The result is sharp with extremal function \( f \) given by
\[
f(z) = \frac{a(1-\gamma)|e^{i\beta}cos\beta|}{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]} z^n.
\]

Proof: It is sufficient to show that
\[
|zf(z) - 1| \leq 1 - \delta , \quad (0 \leq \delta < 1)
\]
for \( |z| < r_1(\beta, \alpha, \gamma, \delta) \). We have
\[
|zf(z) - 1| = \left| \frac{zf(z) - f(z)}{f(z)} \right| \leq \frac{\sum_{n=2}^{\infty} a_n (n-1) z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \leq \sum_{n=2}^{\infty} a_n (n-1) |z|^{n-1}
\]
\[
\leq 1.
\]

Hence, by Theorem (1), (23) will be true if
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1},
\]
or equivalently
\[
|zf(z) - 1| \leq \frac{\sum_{n=2}^{\infty} (n-\delta) a_n |z|^{n-1}}{(1-\delta)} \leq \frac{\sum_{n=2}^{\infty} (n-\delta) a_n |z|^{n-1} - 1}{(1-\delta)} \leq \sum_{n=2}^{\infty} a_n |z|^{n-1}.
\]

Hence, by Theorem (1), (23) will be true if
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

Theorem (6): If \( f \in \mathcal{R}_\delta(\beta, \alpha, \gamma) \), then \( f \) is convex of order \( \delta \) \( (0 \leq \delta < 1) \) in the disk \( |z| < r_2(\beta, \alpha, \gamma, \delta) \), where
\[
r_2(\beta, \alpha, \gamma, \delta) = \text{inf}\left\{ \frac{(1-\delta)n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{(n-\delta)\alpha(1-\gamma)|e^{i\beta}cos\beta|} \right\}^{\frac{1}{n-1}}, n \geq 2. \quad (24)
\]

The result is sharp with extremal function \( f \) given by (22).

Proof: It is sufficient to show that
\[
|zf(z)| \leq 1 - \delta , \quad (0 \leq \delta < 1)
\]
for \( |z| < r_2(\beta, \alpha, \gamma, \delta) \). We have
\[
|zf(z)| = \frac{\sum_{n=2}^{\infty} a_n n |z|^{n-1}}{1 + \sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq \sum_{n=2}^{\infty} a_n n |z|^{n-1}
\]
\[
\leq 1.
\]

Hence, by Theorem (1), (25) will be true if
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1}, \quad n \geq 2. \quad (26)
\]

The result is sharp with extremal function \( f \) given by (22).

Proof: It is sufficient to show that
\[
|f(z) - 1| \leq 1 - \delta , \quad (0 \leq \delta < 1)
\]
for \( |z| < r_3(\beta, \alpha, \gamma, \delta) \). We have
\[
|f(z) - 1| = \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

The last expression above is bounded by \( (1-\delta) \) if
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

Hence, by Theorem (1), (27) will be true if
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1},
\]
or equivalently,
\[
|zf(z) - 1| \leq \frac{n[(n-1)(1+\alpha) + a(1-\gamma)|e^{i\beta}cos\beta|]}{a(1-\gamma)|e^{i\beta}cos\beta|} \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

Setting \( |z| = r_3(\beta, \alpha, \gamma, \delta) \), we get the desired result.
In the following Theorem, we prove the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \) is closed under linear combination.

**Theorem (8):** Let the function \( f_j(z) \in \mathcal{R}_\delta(\beta, \alpha, \gamma) \) defined by
\[
f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0, j = 1, 2, 3, \ldots, l).
\]
Then, we have
\[
h(z) = \sum_{j=1}^{l} c_j f_j(z),
\]
is in the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \), where \( \sum_{j=1}^{l} c_j = 1, \ c_j \geq 0 \).

**Proof:** By definition of \( h(z) \), we have
\[
h(z) = \sum_{j=1}^{l} c_j z - \sum_{n=2}^{l} \sum_{j=1}^{l} c_j a_{n,j} z^n.
\]
Further, since \( f_j(z) \) are in the class \( \mathcal{R}_\delta(\beta, \alpha, \gamma) \) for every \( j = 1, 2, 3, \ldots, l \).

Hence, we can see that
\[
\sum_{n=2}^{\infty} n [(n - 1)(1 + \alpha) + \alpha(1 - \gamma)] e^{i\beta}\cos \beta] \sum_{j=1}^{l} c_j a_{n,j}
\]
\[
= \sum_{j=1}^{l} c_j \sum_{n=2}^{\infty} n [(n - 1)(1 + \alpha) + \alpha(1 - \gamma)] e^{i\beta}\cos \beta] a_{n,j}
\]
\[
\leq (1 - \gamma) e^{i\beta}\cos \beta\sum_{j=1}^{l} c_j = (1 - \gamma) e^{i\beta}\cos \beta].
\]
Let \( f \in \mathcal{R} \) be a function of the form (1). Motivated by Silverman [3] and Silvia [5], see also [4], [6], we define the partial sums \( f_m \) defined by
\[
f_m(z) = z - \sum_{n=2}^{m} a_n z^n, \quad (m \in N).
\]

**Theorem (9):** Let \( f \in \mathcal{R} \) be given by (1) and define the partial sums \( f_j(z) \) and \( f_m(z) \) as follows:
\[
f_m(z) = z - \sum_{n=2}^{m} a_n z^n, \quad (m > 2).
\]

Also suppose that
\[
\sum_{n=2}^{\infty} d_n a_n
\]
\[
\leq 1,
\]
\[
= \frac{n[(n - 1)(1 + \alpha) + \alpha(1 - \gamma)] e^{i\beta}\cos \beta]}{(1 - \gamma) e^{i\beta}\cos \beta}.
\]
Then, we have
\[
\text{Re} \left( \frac{f(z)}{d_m(z)} \right) > \frac{1}{d_m},
\]
and
\[
\text{Re} \left( \frac{f_m(z)}{f(z)} \right) > \frac{d_{m+1}}{1 + d_{m+1}}.
\]
Each of the bounds in (32) and (33) is the best possible for \( m \in N \).

**Proof:** For the coefficients \( d_n \) given by (31), it is not difficult to verify that
\[
d_{n+1} > d_n > 1, \ n = 2, 3, \ldots.\]
Therefore, we have
\[
\sum_{n=2}^{m} a_n + d_m \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n
\]
\[
\leq 1.
\]
By setting
\[
g_1(z) = d_m \frac{f(z)}{f_m(z)} - \left( 1 - \frac{1}{d_m} \right)
\]
and making use of (34), we deduce that
\[
\frac{g_1(z) - 1}{g_1(z) + 1} \leq \frac{d_m \sum_{n=m+1}^{\infty} a_n}{1 - 2 \sum_{n=2}^{m} a_n - d_m \sum_{n=m+1}^{\infty} a_n} \leq 1, \ (z \in U).
\]

which readily yields the left assertion (32). If we take
\[
f(z) = z - \frac{z^m}{d_m},
\]
then
\[
\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{d_m} \rightarrow 1 - \frac{1}{d_m} (z \rightarrow 1),
\]
Similarly, if we take
\[
g_2(z) = (1 + d_m) \frac{f_m(z)}{f(z)} - \frac{d_m}{1 + d_m}
\]
and making use of (34), we deduce that
\[
\frac{g_2(z) - 1}{g_2(z) + 1} \leq \frac{(1 + d_{m+1}) \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^{m} a_n + (1 - d_m) \sum_{n=m+1}^{\infty} a_n} \leq 1,
\]
which leads us to the assertion (33). The bound in (33) is sharp for each \( m \in N \) with the function given by (36).

**References**


