Zhang’s Second Order Traffic Flow Model and Its Application to the Kisii-Kisumu Highway within Kisii County

Jared Nyaberi Bosire¹, Prof. Johana K. Sigey², Dr. Jeconia A. Okelo³, Dr. James Okwoyo⁴

¹,²,³ Department of Pure and Applied Mathematics, JLUAT Kisii CBD, P.O Box 62000 -00200 KISII, Kenya
⁴School of Mathematics University of Nairobi, Kenya

Abstract: Most highways experience traffic jams as a result of various reasons and hence need for a practical solution. Traffic flow models play an important role in both today’s traffic research and in many traffic applications such as traffic flow prediction, incident detection and traffic control. In this project report, I considered the macroscopic Zhang’s second order traffic flow model. Macroscopic modeling approach is most suitable for a correct description of traffic flow. Macroscopic traffic flow variables like traffic density and average traffic velocity, which reflect the average state of the road traffic, were considered. It is the variables that define the order of any traffic flow model. The Zhang’s second-order macroscopic traffic flow model was applied to the Kisii-Kisumu highway within Kisii County. The viscous continuum traffic flow model was studied numerically using the finite difference approximation method. The flow problem was solved to show that it was stable and efficient. The graphical presentation of numerical results produced verified some well understood qualitative behavior of traffic flow. This model was tested using the Kisii-Kisumu highway and may be extended to other roads.

Keywords: Macroscopic flow, Traffic Flow, Velocity, Zhang, Traffic density.

1. Introduction

Traffic congestion on motorways is becoming a pressing problem all over the world due to growing populations. Kenya being one of them. This is as a result of rapidly increasing population and the crowding of motorized traffic onto limited road networks. It can also be attributed to traffic breakdown in initially free flowing traffic. On the Kisii-Kisumu road for instance, apart from vehicles; motorcycles have become a major concern. Their sudden upsurge on the road has made smooth flow of traffic almost impossible. Nowadays traffic flow and traffic density has become a major problem in the society. Traffic jams do not only cause considerable costs due to unproductive time losses; they also augment the probability of accidents and have a negative impact on the environment (air pollution, fuel lost, health problems, noise, stress). One approach to ease congestion is to increase the capacity of existing roadways by addition of lanes Kimathi (2012). This approach is long term, very costly and often not feasible due to environmental and/or societal constraints. Another approach is metering the rate at which vehicles enter a road network, Gonzales et al (2009). This results in need for a short-term solution that involves controlling traffic in such a way that congestion is solved, reduced or at least delayed.

Traffic flow models have been developed to address this problem. They can be used to simulate traffic, for instance, to evaluate the use of a new part of the infrastructure. Research on the subject of traffic flow modeling started when Lighthill and Whitham (1955) presented a model;

$$\rho + (\rho v)_x = 0 \quad (1.1)$$

The aim of traffic flow analysis is to create and implement a model which would enable vehicles to reach their destination in the shortest possible time using the maximum roadway capacity.

2. Literature Review

Different types of traffic flows are described by different models. For equilibrium link flow, the celebrated Lighthill-Witham-Richards (LWR) model was developed in 1956. They separately developed the first dynamics traffic flow model. The LWR model describes traffic using a conservation law where they assumed that the traffic flow is related to the traffic density.

The LWR model is a first-order model in the sense of a PDE system order. Newell (1993) improved the LWR model so as to cope with Shockwaves and stop-and-go traffic in congested traffic situations. Payne (1971) proposed the first continuum traffic flow model. The macroscopic Payne model is of second order since it has two variables: traffic density and average traffic velocity.

Macroscopic flow variables, such as flow, density, speed and speed variance, reflect the average state of the traffic flow in contrast to the microscopic traffic flow variables, which focus on individual drivers.

Helbing (1997) proposed a third-order macroscopic traffic flow model with the traffic density, the average velocity and the variance on the velocity as variables. A great deal of work has been devoted to the study of traffic flow by improvement of already existing models through using various numerical approximation techniques in an attempt to give more accurate results Hoogendoorn and Bovy (2006).The manuscript by Lighthill and Whitham (1955) set the tone for many researchers’ investigations into the theory of traffic flow especially for traffic flows on a single, long, and rather idealized road. Verification of traffic flow models...
Traffic models can be classified according to the level and different factors, which detail traffic flow. Models may be categorized using various phenomena during the rush hours on the Kisii-Kisumu highway in Kisii County. The aim was to obtain in-depth knowledge of the Paramics model for freeway applications. Kotsialos et al. (1999) showed that macroscopic traffic flow models are suitable for large-scale simulation of traffic flow in networks.

A new higher-order continuum model has been developed by Zhan’g (2000).

\[ \rho_t + (\rho v)_x = 0 \]  
\[ v_t + [v + 2\beta c(\rho)]_x + \frac{c^2(\rho)}{\rho^2} \rho_x = \frac{\nu_c}{\tau} + \mu(\rho)_{xx} \]  
where

\[ \mu(\rho) = 2\beta v c(\rho) \]  
\[ c(\rho) = \rho v c(\rho) \]  
\[ \frac{c^2(\rho)}{\rho} \]  
\( \rho_x \) is the anticipation term that describes the response of macroscopic driver to traffic density i.e the space concentration and pressure.

In Kenya, studies have been carried out on traffic flow models on various roads. An example is the Zhang’s second order traffic flow model and its application to the Nairobi-Thika highway. Sigey and Kimathi (2006). They analyzed the wave phenomenon of the highway traffic flow during rush hours. A 3- phase traffic theory has also been developed to explain traffic breakdown and the resulting spatiotemporal features of congested vehicular traffic, Kimathi (2012). In this study, the Zhang’s second-order macroscopic traffic flow model was considered to investigate the traffic flow phenomena during the rush hours on the Kisii-Kisumu highway in Kisii County.

2.1 Classification of traffic flow models

Traffic models can be classified according to the level and detail traffic flow. Models may be categorized using various dimensions (deterministic or stochastic, continuous or discrete, analytical or simulation etc). The most common classification is the distinction between microscopic and macroscopic traffic flow modelling approaches. However, this distinction is not unambiguous, owing to the existence of hybrid models. This is why models are categorized based on either representation where traffic is either macroscopic or microscopic. The observed behaviour of drivers, that is, headways, driving speeds and driving lane, is influenced by different factors, which can be related to the driver–vehicle combination (vehicle characteristics, driver experience, age, gender and so forth), the traffic conditions (average speeds, densities), infrastructure Conditions (road conditions) and external situational influences (weather, driving regulations). These models are based on the analogy between compressible flow in a Navier-Stokes fluid and traffic flow equations Maria R et al (2012) Over the years, different theories have been proposed to (dynamically) relate the observed driving behaviour to the Parameters describing these conditions. Analysts approach the problem in many main ways, corresponding to the main scales of observation in physics.

2.1.1 Macroscopic traffic flow models

A macroscopic traffic model is a mathematical model that formulates the relationship among traffic flow characteristics like density, flow, mean, speed of traffic system. Such models are conventionally arrived at by integrating microscopic traffic flow models and converting the single entity level characteristics to comparable system level characteristics.

A macroscopic model may assume that the traffic stream is properly allocated to the roadway lanes, and employ an approximation to this end. The method of modeling traffic flow at macroscopic level originated under an assumption that traffic streams as whole are comparable to fluid streams. Vehicular motion is however different from the motion of elementary particles. For instance, vehicular motion is directed towards a single direction. Another difference is that unlike in particles where the influence of propagation is strongly biased in one direction, vehicular interaction is strongly influenced by a vehicle right in front of it.

In this research macroscopic traffic flow model has been studied both theoretically and numerically. The study has considered Zhang’s second-order traffic flow model developed by Zhang (1998) and in Zhang (2000) where he developed a finite difference scheme for this model. It has been applied on the Kisii-Kisumu highway in Kisii County. The Zhang model is a macroscopic traffic flow model.

2.2 The fundamental diagram on traffic flow

In understanding the relation between the traffic flow and the traffic density at a given location of the highway, a fundamental diagram is essential. The fundamental diagram (FD) of traffic flow is a basic tool in traffic engineering to understand the flow capacity of a roadway, Shima (2013). The diagram is a macroscopic traffic model that describes a statistical relation between the macroscopic traffic flow variables of flow and density. It provides a graphical depiction of the flow of vehicles along the highway over time. The traffic on the highway is observed and the traffic flow (vehicles/hour) versus the traffic density (vehicles/kilometer/lane) is plotted for a location along the highway. A curve as shown below occurs.

![Figure 1: The Fundamental Diagram](image-url)
3. Methodology

The highway used in this study is a one-way road, with length \( 0 \leq x \leq L \).

Figure 2.1 A section of the one-way road

The assumption before the study was conducted is that vehicles do not appear or disappear. Therefore the number of vehicles will depend only on the number already present in the system and the flow of the vehicles into and out of the system. The Zhang’s model is of second-order as it is comprised of two equations in density \( \rho(x, t) \) and velocity \( v(x, t) \).

3.1 The governing equations.

The wave model of traffic flow theory is the simplest dynamic traffic flow model that reproduces the propagation of traffic waves. It is made up of the continuity equation, the conservation of momentum equation and the initial and boundary conditions.

3.1.1 The continuity equation.

An important relation in traffic flow theory is the continuity equation. This equation is used to relate the instantaneous characteristic of density to the local characteristic flow, (Zhan’g 1998).

The density measured in car length can vary from 0 on an empty highway to 1 in a bumper to bumper traffic. For a section of a highway with an entry point \( p_1 \) and an exit point \( p_2 \) we let \( c_1 \) be the number of cars passing \( p_1 \) in time \( \Delta t \) and \( q_1 \) be the flow.

We also let \( c_2 \) be the number of vehicles passing \( p_2 \) in the time \( \Delta t \) and \( q_2 \) be the flow, (Tom v. 2014).

Then assuming \( c_1 > c_2 \) then it follows that

\[
q_1 = \frac{c_1}{\Delta t}, \quad q_2 = \frac{c_2}{\Delta t}
\]

Then

\[
\Delta q = q_2 - q_1 = \frac{c_1 - c_2}{\Delta t} = -\frac{\Delta c}{\Delta t}
\]

Therefore

\[
\Delta c = -\Delta q \Delta t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (a)
\]

Similarly for

\[
p_1 > p_2, \Delta p = p_2 - p_1 = \frac{\Delta c}{\Delta x}
\]

This implies that

\[
\Delta c = \Delta p \Delta x \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (b)
\]

From (a) and (b) above we get

\[
\Delta p \Delta x + \Delta q \Delta t = 0
\]

Dividing both sides by \( \Delta t \Delta x \),

We get

This culminates to

\[
\frac{\Delta p}{\Delta t} + \frac{\Delta q}{\Delta x} = 0 \quad (2.1)
\]

It is a mathematical representation of the principle of mass conservation i.e. the amount of flux entering and leaving a given channel/point is the same. A basic conjecture of the simple continuum model is that vehicles are not created or lost along the road. \( \rho \) is the fluid density and \( q \) is fluid flow. Vehicles on the highway are thought as compressible fluids.

3.1.2 Conservation of momentum equation.

This equation relates the sum of the forces acting on an element of fluid to its acceleration. It represents the principle of conservation of motion that states that the rate of change of momentum is equal to the net forces. The equation of velocity dynamics can be written in the generalized form as:

\[
v_t - v v_x = \frac{V_r(\rho) - v}{\tau} - \frac{c(\rho)}{\rho - \rho_s}
\]

Where \( \tau \) is the relaxation time and \( V_r \) is equilibrium velocity

\[
V_r(\rho) - V
\]

is the relaxation or acceleration term. This term forces vehicle velocity towards equilibrium. \( C(\rho) \) is an anticipation term which Payne(1971) suggested should be of the form \( C(\rho) = \delta (\rho \theta(\rho)) \) and Zhang (1998) suggesting that the local sound speed be \( c = c_s \). When the relaxation term in equation 2.2 is zero, then the Zhang’s model reduces to the LWR model. Density was measured in units of vehicle/meter with a maximum of \( \rho_{max} = 0.2 \text{veh/m} \) basing the average vehicle for length at 5m. Velocity was measured in units of meters/second; with a maximum of 30m/s. He further included a viscosity term to smooth out sudden density and velocity changes deemed unrealistic.

As modified by Zhang (1998) the model is governed by the equation:

\[
v_t + v v_x = \frac{v_x(\rho v) + c^2}{\rho} + \mu \frac{v_x}{\rho} \quad (2.2)
\]

Where \( \mu \) is the viscosity or dissipative constant.

\[
v_x = \left[v + 2\beta c(\rho) \right] v_x
\]

\[
\mu(\rho) = 2\beta \tau c^2(\rho)
\]

\[
c(\rho) = \rho v(\rho).
\]

Flow of traffic was analysed using equation 2.2 and 2.3.

3.2 Initial and Boundary Conditions

The initial conditions must be defined to solve a problem using this model. A boundary is defined to be \( \rho(x, t) \), representing density as a function of time and position. These boundaries typically take two different forms resulting in the initial value problem (IVP) and the boundary value problem (BVP). The initial value problem gives the traffic densities at time \( t=0 \), such that \( \rho(x, 0) = \rho_1(x) \) where \( \rho_1(x) \) is the given density function similarly velocity at times \( t=0 \) is given by \( v(x, 0) = v_1(\rho_1(x)) \)

Boundary value problem gives some functions \( \dot{\rho}(t) \) such that

\[
\rho(0, t) = \dot{\rho}(t)
\]

Similarly Velocity is given by \( v(0, t) = v_1(\dot{\rho}(t)) \)

The initial conditions are

\[
\rho(x, 0) = \rho(x) \quad (2.5)
\]

\[
v(x, 0) = v_1(\rho(x)) \quad (2.6)
\]

And the boundary conditions are
The input density in this case was
values of the input parameters are listed in table 3.1.

This indicates a concentrated mass of high density traffic.

The initial density profile was given by
features encountered when a section of high density traffic or
a piece of roadway i.e. an interval (a,b) of four sections

And

So that

Equations 2.9 and 2.10 are discreticized to yield

The next chapter is a summary of the results obtained. The
Zhang’s second order model is discussed, and its analytical
and numerical solutions developed.

4. Results and Discussions

To test the model, a section of the highway was selected. The selected section was 500m long from DarajaMbilili market
towards the Kisii-Migori junction and was partitioned as a piece of roadway i.e. an interval (a,b) of four sections
i.e L = 500m and n = 4 . At each zone, the average of
density and velocity over time were computed.

The initial density profile was given by

The input density in this case was

This indicates a concentrated mass of high density traffic.

The goal of this input density was to identify the basic features encountered when a section of high density traffic or
a section of low density traffic enters the roadway. The
values of the input parameters are listed in table 3.1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>1</td>
<td>dimensionless</td>
</tr>
<tr>
<td>(\tau)</td>
<td>0.1</td>
<td>S</td>
</tr>
</tbody>
</table>

The wave solutions are shown in the figures below.

4.1 Early Time Traffic Behaviour

The early time densities and velocities of the incoming traffic with respect to different positions on the highway
were observed. The results are as shown in figures 3.1 and
3.2. The road was divided into four sections which acted as
sensor points i.e. 0m to 100m, 100m to 200m, 200m to
300m and finally 300m to 500m. In the four sections, the
observations were made every minute within four minutes.
The density and velocity of traffic were observed during the
same intervals. At t= 0 min, observations on the initial
cluster of traffic reveals a density of 0.3 veh/m, indicating a
bumper to bumper scenario. The corresponding velocity is
24m/s, a minute later, the density slightly drops with velocity
increasing to 25m/s. The situation remains the same up to
100m position. As the flow approaches 150m, the density
increases to 0.4 veh/m, with the velocity dropping to 20m/s.
the traffic situation is again bumper to bumper. This can be
attributed to the car-following behaviour of the vehicles
behind the initial cluster. As the initial cluster begins to
spread and as the flow approaches 200m, the traffic ahead
sees clearer and increases its speed. Space is created for the
vehicles behind this cluster to equally increase their speed.
Consequently, the density is now oscillating around 0.35
veh/m. The velocity is between 22m/s and 24m/s. The slight
fluctuations in density and velocity at these different times
are minimal though. As the flow reaches 300m, the density
now completely reduces to 0.3 veh/m with the velocity

Figure 3.1: Early Time (density)

Figure 3.2: Early Time (Velocity)
increasing and converging to 25 m/s. From the 300m position, and with clear vision, the density is maintained at 0.3veh/m with vehicular velocity maintaining at 25m/s up to the 500m position. The input density becomes low and traffic simply propagates for early time.

After 20 minutes, the initial cluster spreads and disperses considerably. As the initial cluster spreads, the bunched input density is allowed to begin to spread and speed up at t=20min.

5. Conclusion

In this study, from a practical point of view, a consideration was made on the possibility of using traffic flow as the state variable for traffic control purpose. A macroscopic traffic flow model within the framework of Zhan’g has been presented. This model was applied on a 500m stretch on the Kisii- Kisumu road. The macroscopic modelling approach to traffic flow applied, reproduced some characteristics of known behaviours of vehicles in a highway. i.e vehicular motion is influenced by a vehicle right in front of it and the vehicle in front is affected rather weakly by the vehicle behind it.

This was clearly the case in this study as shown in both early and late time cases. In both cases, flow rate (veh/unit time), and density (veh/unit distance) had an inverse relationship as shown in figures 3.1 and 3.2 for early time behaviours and figures 3.2 and 3.3 for late time behaviours. The model was studied and implemented numerically using a system of equations i.e the continuity equation and the conservation of momentum equation. The numerical method for solving the macroscopic model in conservative form was discussed and tests carried out to show the effectiveness of the method. It is noted that the flow in highways is dictated by the conditions upstream and downstream. Velocity of the traffic therefore depends on the density on the road.

This report builds a solid base for modelling traffic flow macroscopically. The method can be extremely promising in that results resemble hypothesized real world results.

6. Recommendations

Modelling of traffic flow is becoming increasingly important. However the development of new realistic traffic flow models is still a challenging task. For example there are few models that accurately capture other traffic dynamics like mergers and diversions, abrupt lane changing, driver-vehicular interaction and many more. It is therefore recommended in this study that, besides developing new models, there is need to validate traffic flow models that have been proposed such as the PW model. Another direction of research in computation may come from the development of parallel algorithms to simulate traffic flow in large networks.

Recommendations are also made that in future ramp modelling should be the next logical focus of similar research. Data will be taken from traffic sensors at the input of the sections on the highway to be studied. The method provided in this research can be extended to solve other higher order traffic flow models like the multi-lane traffic simulations.

For traffic flow at aggregate level, higher order traffic flow models may be combined in order to capture more dynamics. For example the PW model and the Zhang’s model can be
combined for link flows. This can address modelling at mergers and divergers.

References


Author Profile

Mr. Jared Nyaberi Bosire was born in 1977 in the District of Kisii, Kenya. He did his undergraduate degree in education at Egerton University, Kenya. He has worked at several government institutions as a teacher before doing his Master’s degree in applied mathematics at Jomo Kenyatta University of Agriculture and Technology in Kenya. He is currently a Lecturer at Gusii Institute of technology. He has keen Interest in traffic dynamics. 

Professor Johana Sigej: is an experienced professor of applied mathematics and the director in Jomo Kenyatta University of Agriculture and Technology Kisii-Campus, Kenya. He holds bachelor degree in mathematics and computer science from Jomo Kenyatta University of Agriculture and Technology and masters in applied mathematics from Jomo Kenyatta University of Agriculture and Technology Kenya. He did his PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology. He has published more than ten papers on both heat transfer and traffic flow.

Dr. Jeconia Abonyo Okelo: is a senior lecturer in applied mathematics at Jomo Kenyatta University of Agriculture and Technology in Kenya. He holds bachelor’s degree in education science. Masters in applied Mathematics and a PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology-Kenya. He is affiliated to Jomo Kenyatta University of Agriculture and Technology-Kenya. He has dependable background in applied mathematics in particular fluid dynamics. He has published more than ten papers on both heat transfer and traffic flow.

Dr. James Mariita Okwoyo: is a senior Lecturer at school of mathematics, University of Nairobi-Kenya. He holds bachelor's degree in education science and a PhD in applied mathematics from Jomo Kenyatta University of Agriculture and Technology-Kenya. He is affiliated to the University of Nairobi and has published more than seven papers both heat transfer and traffic flow.