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# The Exquisite Integer Additive Set-Labeling of Graphs

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**Abstract:** ***Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An integer additive set-indexer (IASI) of a graph $G$ is an injective function $f: V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+: E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective, where $f(u) + f(v)$ is the sum set of $f(u)$ and $f(v)$. If $f^+(uv) = k \; \forall \; uv \in E(G)$, then $f$ is said to be a $k$-uniform integer additive set-indexer. In this paper, we study the admissibility of a particular type of integer additive set-indexers by certain graphs.***

**Keywords:** Integer additive set-labeling, integer additive set-indexers, weak integer additive set-labeling, strong integer additive set-labeling, exquisite integer additive set-labeling.

**Mathematics Subject Classification**: 05C78

## 1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [3] and [8] and for different graph classes, we further refer to [4] and [5]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The *sum set* of two non-empty sets $A$and $B$, denoted by $A + B$, is defined as $A + B = \{a + b: a \in A, b \in B\}$. If either $A$ or $B$ is countably infinite, then their sum set $A + B$ is also countably infinite. Hence, all sets we mention here are finite. Using the concepts of sum sets, an integer additive set-labeling of a given graph $G$ is defined as follows.

**Definition 1.1.** [6] Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An *integer additive set-labeling* (IASL, in short) of a graph $G$ is defined as an injective function $f: V(G) \to \mathcal{P}(athbbN_0)$ which induces a function $f^+: E(G) \to \mathcal{P}(\mathbb{N}_0)$ such that $f^+(uv) = f(u) + f(v), uv \in E(G)$. A Graph which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

The notion of an integer additive set-indexers of graphs was introduced in [6].

**Definition 1.2.** [6] An integer additive set-labeling $f: V(G) \to \mathcal{P}(\mathbb{N}_0)$ of a graph $G$ is said to be an *integer additive set-indexer* (IASI) if the induced function $f^+: E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A Graph which admits an IASI is called an *integer additive set-indexed graph* (IASI-graph).

An IASL (or IASI) is said to be $k$-*uniform* if $|f^+(e)| = k$ for all $e \in E(G)$. That is, a connected graph $G$ is said to have a $k$-uniform IASL (or IASI) if all of its edges have the same set-indexing number $k$.

The existence of an integer additive set-labeling (or integer additive set-indexers) by given graphs was established in [12] and the admissibility of integer additive set-labeling (or integer additive set-indexers) by given graph operations and graph products was established in [14].

**Theorem 1.3.** [12] *Every graph $G$ admits an integer additive set-labeling (integer additive set-indexer).*

The number of elements in the set-label of an element of a graph $G$ is called the *set-indexing number* of that element. An element of $G$ having a singleton set-label is called a *mono-indexed element*of $G$.

The IASLs and IASIs, with respect to which the edges of given graphs attain the minimum and maximum possible set-indexing numbers are of special interest. Hence, we have introduced the following notions.

**Definition 1.4.** [7] A *weak integer additive set-labeling* (WIASL) of a graph $G$ is an IASL $f$ such that $|f^+(uv)| = \max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$.

The following is a necessary and sufficient condition for an IASL (or IASI) to be a WIASL (or WIASI) of a given graph $G$.

**Lemma 1.5.** [7] *Let $f$ be an IASL defined on a given graph $G$. Then, $f$ is a WIASL of $G$ if and only if at least one end vertex of every edge of $G$ is mono-indexed.*

**Definition 1.6.** [13] A *strong integer additive set-labeling* (SIASL) of $G$ is an IASL such that if $|f^+(uv)| = |f(u)|\,|f(v)|$ for all $u, v \in V(G)$.
The *difference set* of a set $A$ is the set of all positive differences between the elements of $A$. The difference set of a set $A$ is denoted by $D_A$.

Then, the following result is a necessary and sufficient condition for an IASL (or IASI) to be a SIASL (or SIASI) of a given graph $G$.

**Lemma 1.7.** [13] Let $f$ be an IASL defined on a given graph $G$. Then, $f$ is a SIASL of $G$ if and only if the difference sets of any two adjacent vertices of $G$ are disjoint.

In this paper, we study the characteristic of graphs which admit a certain type of Integer additive set-labeling.

## 2. New Results

First, we introduce the notion of an exquisite integer additive set-labeling as follows.

**Definition 2.1.** Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. An *exquisite integer additive set-labeling* (EIASL, in short) is defined as an integer additive set-labeling $f: V(G) \to \mathcal{P}(\mathbb{N}_0)$ with the induced function $f^+: E(G) \to \mathcal{P}(\mathbb{N}_0)$which is defined by$f^+(uv) = f(u) + f(v), uv \in E(G)$, such that $f(u), f(v) \subseteq f^+(uv)$ for all adjacent vertices $u, v \in V(G)$. A Graph which admits an exquisite integer additive set-labeling is called an *exquisite integer additive set-labeled graph* (EIASL-graph).

An exquisite IASL is called a *weakly exquisite IASL* (WEIASL, in short) if it is a weak IASL and is called a *strongly exquisite IASL* (SEIASL, in short) if it is a strong IASL.

The following theorem is a necessary and sufficient condition for an IASL of a graph $G$ to be an EIASL of $G$.

**Theorem 2.2.***Let $f$ be an IASL of a given graph $G$. Then, $f$ is an EIASL of $G$ if and only if 0 is an element in the set-label of every vertex in $G$.*

***Proof***. Let $f$ be an EIASL of $G$. Then, for two adjacent vertices $u, v$in $G$, we have $f(u), f(v) \subseteq f^+(uv)\ \forall\, u, v \in V(G)$. If 0 is contained neither in $f(u)$ nor in $f(v)$, then some of the smaller element in both the sets will not be in $f^+(uv)$. Hence, assume that exactly one of these two sets, say $f(u)$, contains the element 0. Then, $0 + x = x, \forall\, x \in f(v)$. Hence, 0 in $f(v)$ can not be an element of $f^+(uv)$. In both cases, we get a contradiction to the fact that $f(u), f(v) \subseteq f^+(uv)$. Hence, both $f(u)$ and $f(v)$ must contain the element 0.

Conversely, assume that 0 is an element in the set-label of every vertex in $G$, under $f$. Then, for every element $x \in f(u)$, $x = x + 0$ and hence belongs to $f(u) + f(v) = f^+(uv)$. That is, $f(u) \subseteq f^+(uv)$. In a similar manner, we can also have $f(v) \subseteq f^+(uv)$. Therefore, $f$ is an EIASL of $G$. □

The following theorem establishes a necessary and sufficient condition for a graph $G$ to admit weakly EIASL.

**Theorem 2.3.** *A graph $G$ admits a weakly exquisite integer additive set-labeling if and only if it is a star.*

***Proof***. Assume that $G$ admits an WIASL, say $f$. Then, since $f$ is a WIASI, by Lemma 5, at least one end vertex of every edge of $G$ is mono-indexed. Since $f$ is also an EIASL of $G$, then by Theorem 2, 0 must be an element of the set-label of every vertex of $G$. Hence, $\{0\}$ will be the set-label of one end vertex of every edge of $G$. Since, $f$ is injective, this is possible only when the end vertex of every edge is unique. That is $G$ is a star graph.

Conversely, assume that $G$ is a star graph $K_{1,n}$. Let $v$ the vertex common to all edges in $G$. Label this vertex by the set $\{0\}$ and label all other vertices by distinct non-singleton sets of non-negative integers. Then, this set-labeling is clearly an EIASL of $G$.

A necessary and sufficient condition for a graph $G$ to admit a SEIASL is established in the following theorem.

**Theorem 2.4.***A graph $G$ admits a strongly exquisite integer additive set-labeling if and only if $D_{f(u)} \cap D_{f(v)} = \emptyset$and $f(u) \cap f(v) = \{0\}$, for any two adjacent vertices $u$ and $v$ in $G$.*

***Proof.*** Let $G$ admits an SEIASL, say $f$ and let $u, v$ be any two adjacent vertices in $G$. Since $f$ is an SIASL, by Lemma 7, the difference sets of the set-labels of adjacent vertices are disjoint. That is, $D_{f(u)}$ and $D_{f(v)}$ are disjoint sets. Since $f$ is an EIASL, by Theorem 2, the set-label of every vertex must contain the element 0. Therefore, 0 is a common element of both $f(u)$and $f(v)$. If $f(u)$ and $f(v)$ contains one common element, say $x$, other than 0, then the difference $x - 0 = x$ belongs to both $D_{f(u)}$ and $D_{f(v)}$, contradicting the fact that $f$ is a strong IASI. Therefore, $f(u) \cap f(v) = \{0\}$.

Conversely, assume that $D_{f(u)} \cap D_{f(v)} = \emptyset$and $f(u) \cap f(v) = \{0\}$, for any two adjacent vertices $u$ and $v$ in $G$. Since $D_{f(u)}$ and $D_{f(v)}$ for any edge $uv$in $G$, $f$ is a strong IASL of $G$. Since $f(u) \cap f(v) = \{0\}$, wehave $\{0\} \subset f(v)\ \forall\, v \in V(G)$. Therefore, $f(u) = f(u) + \{0\} \subseteq f(u) + f(v)$. In a similar way, we can prove that $f(v) \subseteq f(u) + f(v)$. Hence, $f$ is an EIASL of $G$. Therefore, $f$ is an SEIASL.

## 3. Operations and Products of EIASL-graphs

Let us first have the following proposition.

**Proposition 3.1.***If $G$ is an EIASL-graph, then any non-trivial subgraph of $G$ also admits an (induced) EIASL.*

Proof of the above theorem follows from the fact that if $f$ is an EIASL defined on a graph $G$, then for a subgraph $H$ of $G$, the restriction function $f|_H$ will be an induced EIASL on $H$.
The following theorem establishes the admissibility of an EIASL by the union of two EIASL-graphs.

**Proposition 3.2.***The union of two EIASL-graphs $G_1$ and $G_2$ admits an EIASL if and only it both $G_1$ and $G_2$ are EIASL-graphs.*

***Proof.*** Let $f_1$ and $f_2$ be the EIASLs of the given graphs $G_1$ and $G_2$ respectively. Then, the function $f$ defined on $G_1 \cup G_2$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V(G_1) \\ f_2(v) & \text{if } v \in V(G_2) \end{cases}$$

is an EIASL on $G_1 \cup G_2$.

Conversely, assume that the graph $G = G_1 + G_2$ admits an EIASL, say $f$. Since both $G_1$ and $G_2$ are subgraphs of $G$, then by Proposition 1, both $G_1$ and $G_2$ admit the induced EIASLs $f|_{G_1}$ and $f|_{G_2}$ respectively.

For two given graphs $G_1$and $G_2$, the join $G_1 + G_2$ is the graph obtained by every vertex of $G_1$ to all vertices of $G_2$. That is, $G_1 + G_2 = G_1 \cup G_2 \cup < E_{ij} >$, where $E_{ij} = \{u_i v_j : u_i \in V(G_1), v_j \in V(G_2)\}$.

**Proposition 3.3.** *The join of two EIASL-graphs is an EIASL graph if and only if both $G_1$ and $G_2$ admit some EIASLs.*

***Proof.*** Let $G_1$ and $G_2$ be two EIASL-graphs whose vertex set are respectively $V(G_1) = \{u_1, u_2, u_3, \dots, u_n\}$and $V(G_2) = \{v_1, v_2, v_3, \dots, v_l\}$. Let $f_1$ and $f_2$ be the EIASLs of the given graphs $G_1$ and $G_2$ respectively. Then, for $1 \le i \ell n, 1 \le j \le l$, define a function $f$ defined on $G_1 \cup G_2$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v = u_i \in V(G_1) \\ f_2(v) & \text{if } v = v_j \in V(G_2) \end{cases}$$

Then, for any edge $e$in $G_1 + G_2$, we have

$$f^+(e) = \begin{cases} f_1(u_i) + f_1(u_j) & \text{if } e = u_i u_j \in E(G_1) \\ f_2(v_r) + f_2(v_s) & \text{if } e = v_r v_s \in E(G_2) \\ f_1(u_i) + f_2(v_j) & \text{if } e = u_i v_j \in E_{ij}. \end{cases}$$

Since $f_1$ and $f_2$ are EIASLs, 0 is an element of $f_1(u_i)$ and $f_2(v_j)$ for all $1 \le i \le n$and $1 \le j \le l$. Hence, $f_1(u_i), f_2(v_j) \subseteq f_1(u_i) + f_2(v_j)$. Hence $f$ is an EIASL of $G_1 + G_2$.

For the converse, assume that the graph $G = G_1 \cup G_2$ admits an EIASL. Then, since both $G_1$ and $G_2$ are subgraphs of $G$, the result follows by Proposition 1. □

Next, let us verify whether the products of EIASL graphs admit (induced) EIASLs. The following theorem establishes the admissibility of an EIASL by the Cartesian product of two EIASL graphs.

**Theorem 3.4.** *The Cartesian product two graphs $G_1$ and $G_2$ admits an EIASL if and only if both $G_1$ and $G_2$ are EIASL-graphs.*

***Proof.*** Let $V(G_1) = \{u_1, u_2, \dots, u_l\}$ and $V(G_2) = \{v_1, v_2, \dots, v_n\}$ and let $f_1$ and $f_2$ be the EIASLs of $G_1$ and $G_2$ respectively.

Let $G = G_1 \square G_2$. Denote the vertex $(u_i, v_j)$ of $G_1 \square G_2$by $w_{ij}$. Define a function $f: V(G) \to \mathcal{P}(\mathbb{N}_0)$by $f(w_{ij}) = f_1(u_i) + f_2(v_j)$. Since $f_1$ and $f_2$ are EIASLs, all set-label $f(w_{ij})$ in $G$ contain the element 0. Hence, by Theorem 2, $f$ is an EIASL of $G$.

To prove the converse part, assume that $G = G_1 \square G_2$ admits an EIASL, say $f$. Clearly, $G$ has $n$ copies of $G_1$ (and $l$ copies of $G_2$). Let $G_{1i}$ be any one of these copies of $G_1$. Clearly, $G_{1i}$ is an isomorphic graph of $G_1$ and is a subgraph of $G$. Then, Proposition 1, $f$ induces an EIASL to the graph $G_{1i}$. Since $G_1$ is isomorphic to $G_{1i}$, the set-labeling of $G_{1i}$, for any $1 \le i \le n$, will be an EIASL for $G_1$ also . Similarly, we can find an EIASLfor $G_2$ also. This completes the proof. □

Using the same argument, we can establish the existence of induced IASIs for the following graph products.

**Theorem 3.5.** *The strong product two graphs $G_1$ and $G_2$ admits an EIASL if and only if both $G_1$ and $G_2$ are EIASL-graphs.*

Another graph product we need to consider in this context is the *corona* of two graphs. The *corona* of two graphs $G_1$ and $G_2$ , denoted by $G_1 \odot G_2$, is the graph obtained by taking $|V(G_1)|$ copies of $G_2$ and joining all vertex of $i$-th copy of $G_2$ to the $i$-th vertex of $G_1$, for all $1 \le i \le |V(G_1)|$. The following theorem establishes the admissibility of EIASL by the corona of two EIASL-graphs.

**Theorem 3.6.** *The corona two graphs $G_1$ and $G_2$ admits an EIASL if and only if both $G_1$ and $G_2$ are EIASL-graphs.*

***Proof.*** Let $f$, $g$ be the EIASLs defined on $G_1$ and $G_2$ respectively. Let $G_{2i}$ be the $i$-th copy of $G_2$in $G_1 \odot G_2$. Let $g_i$ be the set-label of $G_{2i}$, where $g_i(v_{ij}) = \{ik : k \in g(v_j)\}$, where $v_{ij}$ is the $j$-vertex in the $i$-th copy of $G_2$. For all $v \in V(G_2)$, $g(v)$ contains 0, all $g_i(v_{ij})$ also contain 0. Now define the function $h: V(G_1 \odot G_2) \to \mathcal{P}(\mathbb{N}_0)$ by

$$h(v) = \begin{cases} f(v) & \text{if } v \in V(G_1) \\ g_i(v) & \text{if } v \in V(G_{2i}) \end{cases}$$

Clearly, every vertex in $G_1 \odot G_2$ contains the element 0. Then, by theorem 2, $h$ is an EIASL of $G$.

Conversely, let $G_1 \odot G_2$ be an EIASL-graph. Since $G_1$ is a subgraph of $G_1 \odot G_2$, $G_1$ admits an induced EIASL. Since $G_2$ is isomorphic to each copy $G_{2i}$, label the vertices of $G_2$ by the same set-labels of the corresponding verticesof any one copy $G_{2i}$, which will be n EIASL for $G_2$ also. This completes the proof.

In the similar way, we can verify the existence of an induced EIASL for the rooted product of two EIASL-graphs.

**Theorem 3.7.** *Let $G_1$ be a graph and $G_2$ be a rooted graph. Then, their rooted product $G_1 \circ G_2$ is an EIASL graph if both $G_1$ and $G_2$ admit some EIASLs.*

***Proof.*** Let $v_1$ ve the root of $G_2$. Take $|V(G_1)|$ copies of $G_2$ and identify the root of $i$-th copy of $G_2$ to the $i$-th vertex of $G_1$ for all $1 \le i \le |V(G_1)|$. Label the vertices other than the roots of each copy of $G_2$ as explained in 6 and preserve the labeling of $G_1$ for the corresponding vertices in $G_1 \odot G_2$. Then, this set-labeling is an EIASL of $G_1 \odot G_2$.

The converse part is similar to that of 6. This completes the proof.

## 4. Associated Graphs of EIASL-Graphs

Since the set-labels of all vertices of an EIASL graph contains the element 0, the set-labels of all edges of $G$ will also contain 0. As a result, we immediately have the following results.

**Proposition 4.1.** *If a graph $G$ admits an EIASI, say $f$, then its line graph $L(G)$ and total graph $T(G)$ admit an EIASL, induced by $f$.*

***Proof.*** Let $G$ be a graph on $n$ vertices and $m$ edges with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Also, let $f$ be an EIASI defined on $G$.

Every edge of $G$ corresponds to a vertex in $L(G)$. Then, let $V((LG)) = \{u_1, u_2, u_3, \ldots, u_m\}$. Define a function $g: V(L(G)) \to \mathcal{P}(\mathbb{N}_0)$by $g(u_r) = f^+(v_i v_j)$, if the vertex $u_r$ in $L(G)$ corresponds to the edge in $v_i v_j$ in $G$, where $1 \le r \le m$ and $1 \le i, j \le n$. Since, $f$ is an EIASI, $f^+$ is injective and hence $g$ is also an injective function. Since, for every edge $e \in E(G)$, the set $f^+(e)$ contains the element 0, by Theorem 2, $g$ is an EIASL of $L(G)$. □

**Proposition 4.2.***If a graph $G$ admits an EIASI, say $f$, then its total graph $T(G)$ admits an EIASL, which is induced by $f$.*

***Proof.*** Let $G$ be a graph on $n$ vertices and $m$ edges with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Also, let $f$ be an EIASI defined on $G$.

Every element (vertices and edges) of $G$ corresponds to a vertex in its total graph $T(G)$. Hence, $V(T(G))$ contains $m + n$ elements. Let $V(T(G)) = \{u_1, u_2, u_3, \ldots, u_l\}$, where $l = m + n$. Hence, Define a function $g: V(T(G)) \to \mathcal{P}(\mathbb{N}_0)$ by

$$g(u_r) = \begin{cases} f(v_i) & \text{if } u_r \sim v_i \in G \\ f^+(e_j) & \text{if } u_r \sim e_j \in G. \end{cases}$$

Since, for every $v \in V(G)$and $e \in E(G)$, the sets $f(v)$ and $f^+(e)$ contain the element 0, by Theorem 2, $g$ is an EIASL of $T(G)$. □

In a similar way, we can prove the following results for certain other associated graphs of given EIASL graphs.

**Proposition 4.3.***If $G$ is an EIASI graph, then its subdivision graphs are also EIASL-graphs.*

***Proof.*** Let an edge $e = uv$ of $G$ be subdivided by introducing a new vertex, say $w$. Then, the edge $uv$ will be replaced by two edges $uw$and $wv$. Repeat this process on every edge of $G$ and denote the resultant graph by $H$. Define a function $g: V(H) \to thcalP(\mathbb{N}_0)$ which preserves the same set-labeling for the vertices common in $G$ and $H$ and assign the same set-labels of the deleted edges to the corresponding newly introduced vertices. Since $f$ and $f^+$ are injective in $G$, $g$ is also injective. More over, all set-labels in $G$ contains the element 0, the set-labels in $H$ also contain 0. Therefore, by Theorem 2, $g$ is an EIASL of $H$. □

In the similar manner, we can prove the following results also.

**Proposition 4.4.***A graph obtained by contracting some edges of an EIASL-graph also admits an EIASL.*

***Proof.*** Let $f$ be an EIASL defined on $G$. Then, the set-labels of all elements of $G$contain 0. If $u$ and $v$ be two adjacent vertices in $G$, let $G'$ be the graph obtained by removing the edge $uv$ and identifying the vertices $u$ and $v$ to form a singlevertex, say $w$. Label the new vertex $w$ by the same set-label of the deleted edge $uv$ and label all other vertices of $G'$ the same set-labels of the corresponding vertices of $G$. Since the set-label of $uv$ in $G$ also contains 0, the set-labels of all vertices in $G'$ also contain 0. Therefore, by Theorem 2, $G'$ admits an EIASL. □

**Proposition 4.5.***A graph that is homeomorphic to an EIASL-graph also admits an EIASL.*

***Proof.*** Let $f$ be an EIASL defined on $G$. Let $v$ be a vertex of $G$ having degree 2 and is not in any triangle in $G$. Let $u, w$ be the two vertices adjacent to $v$in $G$. Then $u$ nd $w$ are non-adjacent vertices. Now, let $H = (G - v) \cup \{uw\}$. Hence, $H$is a graph homeomorphic to $G$. Since $V(H) \subset V(G)$, the set-labels of all vertices of $H$, induced by the EIASL $f$ of $G$, contains the element 0. Therefore, $f|_H$ is an EIASL of $H$. □

## 5. EIASLs of Graphs with Finite Ground Sets

In this section, we discuss the existence of EIASLs for given graphs with respect to a finite ground set $X \subset \mathbb{N}_0$ instead of the whole countably infinite set $\mathbb{N}_0$. In this type IASLs, the elements of $G$ have distinct subsets of $X$ as their set-labels.

The cardinality of the set-labels and the cardinality of the finite ground set $X$ are of the special interest. The *set-indexing number* of a graph $G$ is the minimum cardinality of the set $X$ which induces a set-labeling for $G$. The set-indexing number of a graph $G$ is denoted by $\sigma(G)$.

Let us now proceed to discuss the characteristics of graphs which admit EIASLs with respect to a finite ground set $X$. The first question we are going to check here is whether the ground set $X$ itself can be a set-label of a vertex of $G$ under an EIASL$f$.

**Proposition 5.1.** *Let $f$ be an EIASL defined on a graph $G$. Then, the ground set $X$ is the set-label of a vertex $v$ of $G$ if $v$ is a pendant vertex, whose adjacent vertex has the set-label* $\{0\}$.

***Proof.*** Note that $\{0\} + X = X$ and for any integer $x > 0$, $X \subsetneq \{x\} + X$. Hence, $v$ can be adjacent only to a vertex, say $u$, which has a set-label $\{0\}$. Therefore, $v$ is a pendant vertex.□

The following result is about the lower bound for the set-indexing number of a graph $G$.

**Theorem 5.2.***Let $G$ be a graph on $n$ vertices. Then, the minimum value of $\sigma(G)$is $1 + \log_2 n$.*

***Proof.*** Let $G$ be a graph which admits an EIASL, say $f$. Then, by Theorem 2, 0 must be an element in the set-labels of all vertices of $G$. That is $X$ must have at least $n$ subsets containing the element $0$. That is, $n \leq 2^{|X|-1}$. That is, $|X| \geq 1 + \log_2 n$. Hence, the lower bound for $\sigma(G)$is $1 + \log_2 n$. □

**Theorem 5.3.***Let $f$ be an EIASL defined on a given graph $G$ with respect to a finite ground set $X$. Then, if $V(G)$ is $l$-uniformly set-indexed, then $n \leq \binom{|X|-1}{l-1}$.*

***Proof.*** Let $G$ be a graph on $n$ vertices, which admits an EIASL, say $f$, with respect to a finite set $X$. For a positive integer $l$, let $V(G)$ is $l$-uniformly set-indexed. Then, by Theorem 2, $X$ must have at least $n$ subsets containing $l$ elements, all of which contain $0$. The number of $l$-element subsets of $X$ containing $0$is $\binom{|X|-1}{l-1}$. Therefore, $n \leq \binom{|X|-1}{l-1}$. □

## 6. Conclusion

In this paper, we have discussed the existence of exquisite integer additive set-labeling for given graphs and certain graphs associated with the given EIASL-graphs. We have also discussed the existence of finding exquisite IASL for the operations and products of two graphs. Some problems in this area are still open. Some of the most promising problems in this area are the following.

**Problem 1.** Determine conditions for an arithmetic IASI of a given graph $G$ to be an EIASL of $G$.

**Problem 2.** Determine conditions for the direct product, lexicographic product and certain other graph products of EIASL graphs to admit EIASLs.

**Problem 3.** Determine the set-indexing number of a graph $G$ so that it admits WEIASI and SEIASI, both uniform and non-uniform.

More properties and characteristics of this type IASL, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain other types IASLs are still open. Studies about those IASLs which assign sets having specific properties, to the elements of a given graph are also noteworthy. All these facts highlight a wide scope for further studies in this area.

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