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Fixed Point Theorem of the Iterated Function Systems Consisting of $\alpha-\varphi$ - Contraction Type Mappings

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Abstract: ***In this paper we present some fixed point theorem of iterated function system consisting of*** $\alpha-\varphi$ ***-contraction type mappings in complete metric space.***

Keywords: Fixed points; $\alpha-\varphi$-contraction mapping; Iterated function systems; α - admissible.

1. Introduction and Preparatory Results

The most well known result in the theory of fixed points is Banach's contraction mapping principle.

Bessem, Calogero and Pasquale [1] proved the theorem of existence of fixed point of an $\alpha-\varphi$-contraction mapping in complete metric space. They discussed the Banach contraction principle with some generalized contraction conditions and weakened the usual contraction condition.

Many researches have studied the fixed point theorem on the complete metric space (X, d), however, there are few results for the existence of fixed point on the complete metric space $(H(X), h)$ with the use of fixed point theorem on (X, d) (see [2, 3, 4]).

The aim of this paper is to obtain the fixed point theorems of the some generalized contractions in complete metric space $(H(X), h)$.

Before we establish the fixed point theorems in metric space $(H(X), h)$, we discuss some basic results.

In [1], Bessem Samet, Calogero Vetro, Pasquale Vetro proved the following results in the complete metric space.

Denote with Φ the family of non-decreasing functions $\varphi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{+\infty}\varphi^n(t)<+\infty$ for each $t>0$, where φ^n the n-th iterate of φ.

Definition 1.1 (see [1]) Let (X, d) be a metric space and $f:X\to X$ be a given mapping. We say that f is an $\alpha-\varphi$-contraction mapping if there exist two functions $\alpha:X\times X\to[0,+\infty)$ and $\varphi\in\Phi$ such that

$$\alpha(x, y)d(f(x), f(y))\le\varphi(d(x, y)),$$

for all $x, y\in X$. If $\alpha(x, y)=1$ for all $x, y\in X$ and $\varphi(t)=kt$ for all $t\ge 0$ and some $k\in[0, 1)$, then $f:X\to X$ satisfies the Banach contraction principle. There is example involving a function f that is not continuous (see [1]).

If $\alpha(x, y)=1$ for all $x, y\in X$ and $\lim_{n\to+\infty}\varphi^n(t)=0$ for all $t>0$ (not necessarily $\sum_{n=1}^{+\infty}\varphi^n(t)<+\infty$ for each $t>0$), then $f:X\to X$ satisfies a condition of the Matkowski's contraction theorem (see [2]).

Definition 1.2 (see [1]) Let $f:X\to X$ and $\alpha:X\times X\to[0,+\infty)$. We say that f is α -admissible if

$$x, y\in X,\ \alpha(x, y)\ge 1\Rightarrow\alpha(f(x), f(y))\ge 1.$$

Theorem 1.1 (see [1]) Let (X, d) be a complete metric space and $f:X\to X$ be an $\alpha-\varphi$ -contraction mapping satisfying the following conditions:

(1) f is α-admissible;
(2) There exists $x_0\in X$ such that $\alpha(x_0, f(x_0))\ge 1$;
(3) f is continuous.

Then, f has a fixed point, that is, there exists $x^*\in X$ such that $f(x^*)=x^*$.

This fixed point theorem extended the results of Banach's contraction principle and Matkowski's fixed point theorem (see [2, 3, 4]).

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2. Main Results

We now give the theorems of fixed point of the some generalized contractions in the complete metric space $(H(X), h)$.

Let (X, d) be a metric space and $H(X)$ the class of all nonempty compact subsets of X.

That is, $H(X)$ denotes the space whose points are the compact subsets of X, other than the empty set.

Define
$d(A,B) = \max\{d(x,B) : x \in A\}$.
$d(A,B)$ is the distance from the set $A \in H(X)$ to the set $B \in H(X)$.

Define $h : H(X) \times H(X) \to [0, +\infty)$ by
$$h(A, B) := \max\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\}$$
, for all $A, B \in H(X)$. The metric space $(H(X), h)$ is complete provided that (X, d) is complete (for details see [2, 3, 4]). The classical iterated function system (IFS) consists of a finite family of Banach contractions on X to itself. Then there is a unique nonempty compact invariant subset of X with respect to these contractions. In what follows, we extend this IFS by considering a family of $\alpha - \varphi$-contractions.

We assume that (X, d) is a complete metric space and $f : X \to X$ is a continuous mapping on the metric space (X, d) such that
$$h(A, B) = d(x^*, y^*) \Leftrightarrow h(f(A), f(B)) = d(f(x^*), f(y^*))$$
for all $A, B \in H(X)$.
Because $f : X \to X$ is a continuous mapping on the metric space (X, d), F_f maps $H(X)$ into itself.
Define $F_f : H(X) \to H(X)$ by $F_f(A) := f(A)$ for all $A \in H(X)$.

Example 2.1

(1) Let $f(x) := 4x(1-x)$, $A := [0, \frac{1}{8}]$ and $B := [\frac{5}{8}, \frac{7}{8}]$. Then $h(A, B) = \frac{3}{4} = d(\frac{1}{8}, \frac{7}{8})$ and
$$h(f(A), f(B)) = \frac{3}{4} = d(\frac{7}{16}, \frac{15}{16}) \neq d(\frac{7}{16}, \frac{7}{16}) = d(f(\frac{1}{8}), f(\frac{7}{8}))$$
.
That is, for some $A, B \in H(X)$,
$$\{(x^*, y^*) \mid h(A, B) = d(x^*, y^*)\} \neq \{(x^*, y^*) \mid h(f(A), f(B)) = d(f(x^*), f(y^*))\}$$
.Let $f(x) := \frac{1}{3}x, x \in [0, +\infty)$. Then
$$h(A, B) = d(x^*, y^*) \Leftrightarrow h(f(A), f(B)) = d(f(x^*), f(y^*))$$
for all $A, B \in H(X)$.
Let $\alpha : X \times X \to [0, +\infty)$ be a function.
We define $\alpha_\alpha : H(X) \times H(X) \to [0, +\infty)$ by
$$\alpha_\alpha(A, B) := \inf\{\alpha(x^*, y^*) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\}$$

If $\alpha(x, y) = 1$ for all $x, y \in X$ then $\alpha_\alpha(A, B) = 1$ for all $A, B \in H(X)$.

Lemma 2.1 If f is α - admissible then F_f is α_α - admissible.

Proof Let $\alpha_\alpha(A, B) \geq 1$.
Since
$$\alpha_\alpha(A, B) = \inf\{\alpha(x^*, y^*) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\} \geq 1$$
,
for all $x^* \in A, y^* \in B$ such that $h(A, B) = d(x^*, y^*)$, $\alpha(x^*, y^*) \geq 1$.
Because f is α-admissible, $\alpha(f(x^*), f(y^*)) \geq 1$.
By definition of f, if $h(A, B) = d(x^*, y^*)$ then $h(f(A), f(B)) = d(f(x^*), f(y^*))$.
Hence
$$\alpha_\alpha(F_f(A), F_f(B)) = \alpha_\alpha(f(A), f(B))$$
$$= \inf\{\alpha(f(x^*), f(y^*)) \mid d(f(x^*), f(y^*)) = h(f(A), f(B)), f(x^*) \in f(X), f(y^*) \in f(Y)\}$$
$$= \inf\{\alpha(f(x^*), f(y^*)) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\}$$
.
For all $x^* \in A, y^* \in B$ such that $h(A, B) = d(x^*, y^*)$, $\alpha(f(x^*), f(y^*)) \geq 1$.
So
$$\inf\{\alpha(f(x^*), f(y^*)) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\} = \alpha_\alpha(F_f(A), F_f(B)) \geq 1$$
.
Therefore, if $A, B \in H(X); \alpha_\alpha(A, B) \geq 1$ then $\alpha_\alpha(F_f(A), F_f(B)) \geq 1$ that is, F_f is α_α-admissible.

Lemma 2.2 If there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$, then there exists $A_0 \in H(X)$ such that
$\alpha_\alpha(A_0, F_f(A_0)) \geq 1$.

Proof Let $A_0 := \{x_0\} \in H(X)$.
$$\alpha_\alpha(A_0, F_f(A_0)) = \alpha_\alpha(A_0, f(A_0))$$
$$= \inf\{\alpha(x_0, f(x_0)) \mid d(x_0, f(x_0)) = h(A_0, f(A_0)), x_0 \in A_0, f(x_0) \in f(A_0)\}$$
$$= \alpha(x_0, f(x_0)) \geq 1.$$
The system $\{X, f_i : i = 1, 2, \cdots, N\}$ consisting of a family of continuous maps $f_i : X \to X$, will be called

iterated function system, shortly IFS on X. We define $F_{f_1, \cdots, f_N} : H(X) \to H(X)$ by

$$F_{f_1, \cdots, f_N}(A) := \bigcup_{i=1}^{N} f_i(A) \text{ for all } A \in H(X).$$

The image of $\phi \neq A \subset X$ under f_i is given by

$$f_i(A) := \bigcup_{x \in A} f_i(x).$$

We consider continuities that are meant in the Hausdorff sense.

Map $F_{f_1, \cdots, f_N} : H(X) \to H(X)$ is

(1) contraction, if there exists a Lipschitz constant $0 \le L < 1$ such that $h(F_{f_1, \cdots, f_N}(A), F_{f_1, \cdots, f_N}(B)) \le L \cdot h(A, B)$ for all $A, B \in H(X)$.

(2) weak contraction, if there exists a comparison function $\varphi \in \Phi$ such that $h(F_{f_1, \cdots, f_N}(A), F_{f_1, \cdots, f_N}(B)) \le \varphi(h(A, B))$ for all $A, B \in H(X)$.

(3) continuous at $H(X)$, if

$$\forall A \in H(X), \forall \varepsilon > 0, \exists \delta(A, \varepsilon) > 0, \forall B \in H(X)(h(A, B) < \delta(A, \varepsilon));$$

$$; h(F_{f_1, \cdots, f_N}(A), F_{f_1, \cdots, f_N}(B)) < \varepsilon.$$

Remark 2.1

1) If f_i for all $i = 1, 2, \cdots, N$ is Banach contraction then $F_{f_1, \cdots, f_N}$ is Banach contraction too (see [3]).

2) If f_i for all $i = 1, 2, \cdots, N$ is weak contraction then $F_{f_1, \cdots, f_N}$ is weak contraction too (see [3]).

Let $\{X, f_i : i = 1, 2, \cdots, N\}$ be a IFS consisting of continuous functions. Now, we give a counter-example involving a map $F_{f_1, \cdots, f_N} : H(X) \to H(X)$ that is not continuous.

Counter-Example: Consider the functions

$$f_1(x) = \begin{cases} 0, & x \le 0 \\ -2x, & x > 0 \end{cases} \text{ and } f_2(x) = \begin{cases} -2x, & x \le 0 \\ 0, & x > 0 \end{cases}.$$

Then $f_1^n(x) = f_2^n(x) = 0$ for all $n \ge 2$ and for all $x \in R$. Moreover, since $f_{1,2}(x) := \bigcup_{i=1}^{2} f_i(x)$, $f_{1,2}(\{x\}) := \bigcup_{x \in \{x\}} f_{1,2}(x)$ and

$$F_{f_1, f_2}(\{x\}) = \bigcup_{i=1}^{2} f_i(\{x\}) = f_{1,2}(\{x\}),$$

$F_{f_1, f_2}(\{x\}) = \{0, -2x\}$, $F^2{}_{f_1, f_2}(\{x\}) = \{0, 4x\}$ and $F^n{}_{f_1, f_2}(\{x\}) = \{0, (-2)^n x\}$ for all $x \neq 0$.

$F_{f_1, f_2}(\{0\}) = \{0\}$, $F^2{}_{f_1, f_2}(\{0\}) = \{0\}$ and $F^n{}_{f_1, f_2}(\{0\}) = \{0\}$.

Since for all fixed $x > 0$, $h(F^{2n}{}_{f_1, f_2}(\{x\}), F^{2n}{}_{f_1, f_2}(\{0\})) = (-2)^{2n} x \to +\infty$ and $h(F^{2n-1}{}_{f_1, f_2}(\{x\}), F^{2n-1}{}_{f_1, f_2}(\{0\})) = (-2)^{2n-1} x \to -\infty$ as $n \to +\infty$, $h(F^n{}_{f_1, f_2}(\{x\}), F^n{}_{f_1, f_2}(\{0\}))$ is not convergence. It is clear that $F_{f_1, \cdots, f_N}$ is not continuous at $\{0\}$. This demonstrates the discontinuity of F_{f_1, f_2}.

So we assume that for all $i = 1, 2, \cdots, N$, $f_i : X \to X$ are continuous mappings such that $F_{f_1, \cdots, f_N} : H(X) \to H(X)$ is a continuous mapping.

Lemma 2.3 $f : X \to X$ be a $\alpha - \varphi$ -contraction mapping. Then

$$\alpha_\alpha(A, B) h(F_f(A), F_f(B)) \le \varphi(h(A, B)).$$

That is, $F_f : H(X) \to H(X)$ is a $\alpha_\alpha - \varphi$ -contraction (with the same function φ).

Proof New let $A, B \in H(X)$. Then

$$\alpha_\alpha(A, B) \cdot d(F_f(A), F_f(B)) = \alpha_\alpha(A, B) \cdot d(f(A), f(B))$$

$$= \alpha_\alpha(A, B) \cdot \max_{f(x) \in f(A)} \min_{f(y) \in f(B)} d(f(x), f(y)) = \alpha_\alpha(A, B) \cdot \max_{x \in A} \min_{y \in B} d(f(x), f(y))$$

$$= \inf\{\alpha(x^*, y^*) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\} \cdot \max_{x \in A} \min_{y \in B} d(f(x), f(y))$$

$$\le \inf\{\alpha(x^*, y^*) \mid d(x^*, y^*) = h(A, B), x^* \in X, y^* \in Y\} \cdot d(f(x^*), f(y^*))$$

$$\le \alpha(x^*, y^*) \cdot d(f(x^*), f(y^*)) \le \varphi(d(x^*, y^*)) \le \varphi(h(A, B))$$

.

Similarly

$$\alpha_\alpha(A, B) \cdot d(F_f(B), F_f(A)) \le \varphi(h(A, B)).$$

Hence

$$\alpha_\alpha(A, B) h(F_f(A), F_f(B))$$

$$= \max\{\alpha_\alpha(A, B) \cdot d(F_f(A), F_f(B)), \alpha_\alpha(A, B) \cdot d(F_f(B), F_f(A))\} \le \varphi(h(A, B))$$

.

This completes the proof.

Theorem 2.1 Let (X, d) be a complete metric space and $f : X \to X$ be an $\alpha - \varphi$ - contraction mapping satisfying the following conditions:

(1) f is α -admissible;

(2) there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \ge 1$;

Then the mapping $F_f : A \to f(A)$ $(A \in H(X))$ is an $\alpha_\alpha - \varphi$ -contraction (with the same function φ) too from

$H(X)$ into itself. That is, F_f has fixed point and there exists $K \in H(X)$ such that
$F_f(K) = K$.

Proof $(H(X), h)$ is a complete metric space (see [2, 3]) and by Lemma 1, Lemma 2 and Lemma 3, $F_f : H(X) \to H(X)$ is an $\alpha_\alpha - \varphi$ -contraction mapping satisfying the following conditions:

(1) F_f is α_α -admissible;

(2) there exists $A_0 \in H(X)$ such that $\alpha(A_0, F_f(A_0)) \geq 1$;

(3) F_f is continuous.

Hence by Theorem 1.1, F_f has a fixed point $F_f(K) = K \in H(X)$.

The next Theorem provides an important method for combining contraction mappings on $(H(X), h)$ to produce new contraction mappings on $(H(X), h)$.

Theorem 2.2 Let (X, d) be a complete metric space and for $i = 1, 2, \cdots, N$, $f_i : X \to X$ be an $\alpha - \varphi$ -contraction mapping satisfying the following conditions:

(1) f_i is α -admissible for all $i = 1, 2, \cdots, N$;

(2) there exists $x_0 \in X$ such that $\alpha(x_0, f_i(x_0)) \geq 1$ for all $i = 1, 2, \cdots, N$.

Then the mapping $F_{f_1, \cdots, f_N}$ is an $\alpha_\alpha - \varphi$ -contraction too from $H(X)$ into itself. That is, $F_{f_1, \cdots, f_N}$ has fixed point $K \in H(X)$ such that
$F_{f_1, \cdots, f_N}(K) = K$.

Proof We demonstrate the claim for $N = 2$. An inductive argument then completes the proof.
Let $A, B \in H(X)$. By Lemma 2.3,
$$\alpha_\alpha(A, B) h(F_{f_1}(A), F_{f_1}(B)) \leq \varphi(h(A, B))$$
and
$$\alpha_\alpha(A, B) h(F_{f_2}(A), F_{f_2}(B)) \leq \varphi(h(A, B)).$$
Because for A, B, C and D, in $H(X)$ $h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$ (see [4]),
$$\alpha_\alpha(A, B) h(F_{f_1, f_2}(A), F_{f_1, f_2}(B)) = \alpha_\alpha(A, B) h(f_1(A) \cup f_2(A), f_1(B) \cup f_2(B))$$
$$\leq \alpha_\alpha(A, B) \max\{h(f_1(A), f_1(B)), h(f_2(A) \cup f_2(B))\}$$
$$\leq \max\{\alpha_\alpha(A, B) h(f_1(A), f_1(B)), \alpha_\alpha(A, B) h(f_2(A) \cup f_2(B))\} \leq \varphi(h(A, B))$$

Hence $F_{f_1, f_2} : H(X) \to H(X)$ is a $\alpha_\alpha - \varphi$ -contraction mapping.
Let $A_0 := \{x_0\} \in H(X)$.
Since
$\alpha_\alpha(A_0, F_{f_1, f_2}(A_0)) = \alpha_\alpha(A_0, f_1(A_0) \cup f_2(A_0))$, $\alpha(x_0, f_1(x_0)) \geq 1$
and $\alpha(x_0, f_2(x_0)) \geq 1, \alpha_\alpha(A_0, F_{f_1, f_2}(A_0)) \geq 1$.
So F_{f_1, f_2} is α_α -admissible (cf. proof of the Lemma 2.1).
Hence the result of theorem follows from the Theorem 1.1.
This completes the proof.

Corollary 2.1 Let (X, d) be a complete metric space and for all $i = 1, 2, \cdots, N$, $f_i : X \to X$ be an $\alpha_i - \varphi_i$ -contraction mapping satisfying the following conditions:

(1) f_i is α_i -admissible for all $i = 1, 2, \cdots, N$;

(2) there exists $x_0 \in X$ such that $\alpha_i(x_0, f_i(x_0)) \geq 1$ for all $i = 1, 2, \cdots, N$.

Then the mapping $F_{f_1, \cdots, f_N}$ is an $\alpha_\alpha - \varphi$ -contraction too from $H(X)$ into itself. That is, $F_{f_1, \cdots, f_N}$ has fixed point $K \in H(X)$ such that
$F_{f_1, \cdots, f_N}(K) = K$, where $\varphi(t) := \max_{1 \leq i \leq N} \varphi_i(t)$ and $\alpha_\alpha(A, B) := \min_{1 \leq i \leq N} \alpha_{\alpha_i}(A, B)$.

Proof Since $\varphi(t) := \max_{1 \leq i \leq N} \varphi_i(t)$ is non-decreasing continuous functions, it is obvious.

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