Zero-Free Region for Polynomials with Restricted Coefficients

G. L. Reddy¹, P. Ramulu², C. Gangadhar³

¹School of Mathematics and statistics, University of Hyderabad, India-500046
²Department of Mathematics, Govt. Degree College, Wanaparthy, Mahabubnagar, Telangana, India 509103
³School of Mathematics and statistics, University of Hyderabad, India-500046

Abstract: In this paper we prove some extension of the Eneström-Kakeya theorem says that. Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \) then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \). By relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Eneström-Kakeya Theorem.

Mathematics Subject Classification: 30C10, 30C15.

Keywords: Zeros of polynomial, Eneström-Kakeya theorem

1. Introduction

The well known Results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following.

Theorem (A₁). Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \) then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \).

Applying the above result to the polynomial \( z^n P(z) \) we get the following result:

Theorem (A₂). If \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_{n-1} \leq a_{n-2} \leq \cdots \leq a_0 \) then \( P(z) \) does not vanish in \( |z| < 1 \).

In the literature [3-10], there exist several extensions and generalizations of the Eneström-Kakeya Theorem.

In this paper we give generalizations of the above mentioned results. In fact we prove the following results:

Theorem 1. Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with real coefficients such that

\[
a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

then (i) all the zeros of \( P(z) \) does not vanish in the disk \( |z| < \frac{|a_0|}{a_0 + |a_0| - a_n + S_1} \) if both \( n \) and \( (n-m) \) are even or odd

where \( S_1 = 2 \left( (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right) \)

(ii) all the zeros of \( P(z) \) does not vanish in the disk \( |z| < \frac{|a_0|}{a_0 + |a_0| - a_n + S_2} \) if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

where \( S_2 = 2 \left( (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) \right) \)

Corollary 1. Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with positive real coefficients such that

\[
a_0 \geq a_1 \geq \cdots \geq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

then (i) all the zeros of \( P(z) \) does not vanish in the disk \( |z| < \frac{a_0}{a_0 + S_1} \) if both \( n \) and \( (n-m) \) are even or odd

(OR)

\[
a_0 \geq a_1 \geq \cdots \geq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

then (ii) all the zeros of \( P(z) \) does not vanish in the disk \( |z| < \frac{a_0}{a_0 + S_2} \) if both \( n \) and \( (n-m) \) are even or odd

where \( S_1 = 2 \left( (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right) \)

(OR)

where \( S_2 = 2 \left( (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) \right) \)
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where \[ S_2 = 2\left( a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1} \right) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) \]

**Remark 1.** By taking \( a_i > 0 \) for \( i = 0, 1, 2, \ldots, n \), in theorem 1, then it reduces to Corollary 1.

**Theorem 2.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with real coefficients such that

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \\
\geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

if both n and (n-m) are even or odd

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \\
\geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0|}{|a_1| a_n - a_{n-m}} \]

if both n and (n-m) are even or odd

where \[ T_1 = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1} \right) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0|}{|a_1| a_n - a_{n-m}} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where \[ T_2 = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1} \right) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \]

**Corollary 2.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with positive real coefficients such that

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \\
\geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

if both n and (n-m) are even or odd

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \\
\geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n
\]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{T_1 - a_0} \]

if both n and (n-m) are even or odd

where \[ T_1 = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1} \right) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{T_2 - a_0} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where \[ T_2 = 2\left( a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m} \right) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \]

**Remark 2.** By taking \( a_i > 0 \) for \( i = 0, 1, 2, \ldots, n \), in theorem 2, then it reduces to Corollary 4.

**Theorem 3.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with real coefficients such that

\[
a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \\
\leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if both n and (n-m) are even or odd

\[
a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \\
\leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{|a_1| a_n + |a_1| U_1} \]

if both n and (n-m) are even or odd

where \[ U_1 = 2\left( a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2} \right) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{|a_1| a_n + |a_1| U_2} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where \[ U_2 = 2\left( a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1} \right) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m}) \]

**Corollary 3.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with positive real coefficients such that

\[
a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \\
\leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if both n and (n-m) are even or odd

\[
a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \\
\leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{a_1 + 2a_n + U_1} \]

if both n and (n-m) are even or odd

where \[ U_1 = 2\left( a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2} \right) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{a_0}{a_1 + 2a_n + U_2} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where

\[ U_2 = 2(\frac{a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}}{a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}}) \]

**Remark 3.** By taking \( a_i > 0 \) for \( i = 0, 1, 2, ..., n \) in theorem 3, then it reduces to Corollary 3.

**Theorem 4.** Let \( P(z) = \sum^n_{i=0} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with real coefficients such that

\[ a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n \]

if both n and (n-m) are even or odd

\( OR \)

\[ a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then

(i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0| + \sum a_i - a_0 + V_2}{2|a_0| + \sum a_i - a_0 + V_2} \]

if both n and (n-m) are even or odd

where

\[ V_1 = 2(\frac{a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}}{a_0 + a_2 + \cdots + a_{n-m-2} + a_{n-m}}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0| + \sum a_i - a_0 + V_2}{2|a_0| + \sum a_i - a_0 + V_2} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

\[ V_2 = 2(\frac{a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1} - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m})}{a_0}) \]

**Corollary 4.** Let \( P(z) = \sum^n_{i=0} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with positive real coefficients such that

\[ a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n \]

if both n and (n-m) are even or odd

\( OR \)

\[ a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then

(i) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0|}{2a_0 - a_2 + V_2} \]

if both n and (n-m) are even or odd

where

\[ V_1 = 2(\frac{a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}}{a_0 + a_2 + \cdots + a_{n-m-2} + a_{n-m}}) \]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[ |z| < \frac{|a_0|}{2a_0 - a_2 + V_2} \]

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

\[ |z| > \frac{1}{a_0}(a_0 + a_n - a_n + S_1) \]

\[ \Rightarrow R(z) > 0 \quad if \quad |z| > \frac{1}{a_0}(a_0 + a_n - a_n + S_1) \]

**Proof of the Theorem 1.**

Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree n

Let us consider the polynomial \( J(z) = z^n P(\frac{1}{z}) \)

and \( R(z) = (z-1)J(z) \) so that

\[ R(z) = (z-1)\left\{ a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{m-1} + a_m z^{m-1} + \cdots + a_{n-1} z + a_n \right\} \]

Also if \( |z| > 1 \), then

\[ \frac{1}{|z|^{n-1}} < |z| < 0, 1, 2, ..., n-1. \]

Now

\[ |R(z)| \geq |a_0| |z|^{n-1} - \left\{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{n-1} - a_n||z|^1 + |a_n| \right\} \]

\[ \geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|}(|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{n-1} - a_n| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n|) \right\} \]

\[ \geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|}(C_0 - a_1 + a_1 - a_2 + a_2 + a_3 + a_4 + \cdots + a_{n-1} - a_n + a_n - a_{n-1} + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n) \right\} \]

\[ \geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|}(C_2 + a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \cdots + a_{n-1} - a_n + a_n - a_{n-1} + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n) \right\} \]

\[ \geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|}(C_2 + a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \cdots + a_{n-1} - a_n + a_n - a_{n-1} + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n) \right\} \]

\[ \geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|}(C_2 + a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \cdots + a_{n-1} - a_n + a_n - a_{n-1} + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n) \right\} \]

\[ \Rightarrow R(z) > 0 \quad if \quad |z| > \frac{1}{|a_0|}(a_0 + a_n - a_n + S_1) \]
This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk
$$|z| \leq \frac{1}{|a_0|} \{a_0 + |a_n| - a_n + S_1\}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in
$$|z| \leq \frac{1}{|a_0|} \{a_0 + |a_n| - a_n + S_1\}$$

Since $P(z) = z^n J_{-\frac{1}{z}}$ it followed by replacing $z$ by $\frac{1}{z}$, all the zeros of $P(z)$ lie in
$$|z| \geq \frac{|a_0|}{a_0 + |a_n| - a_n + S_1},$$

if both $n$ and $(n-m)$ are even or odd.

Hence all the zeros of $P(z)$ does not vanish in the disk
$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + S_2}$$

if both $n$ and $(n-m)$ are even or odd where
$$S_2 = 2\{a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}\} - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})$$

Similarly we can also prove for if $n$ is even and $(n-m)$ is odd (or) if $n$ is odd and $(n-m)$ is even. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros of $P(z)$ does not vanish in the disk.

This completes the proof of the Theorem 1.

**Proof of the Theorem 2.**

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n$

Let us consider the polynomial $J(z) = z^n P(\frac{1}{z})$

and $R(z) = (z - 1)J(z)$ so that

$$R(z) = (z - 1) \{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_n z + a_n\}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < f or i = 0, 1, 2, \ldots, n - 1$.

Now

$$|R(z)| \geq |a_0||z|^{n+1} - \{|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_n - a_n||z| + |a_n|\}$$

$$\geq |a_0||z|^{n+1} \{ |z| - \frac{|a_0 - a_1|}{|a_0|} + \frac{|a_2 - a_3|}{|a_0|^2} + \frac{|a_3 - a_4|}{|a_0|^3} + \cdots + \frac{|a_{m-1} - a_m|}{|a_0|^{n-m+1}} + \frac{|a_m - a_{m+1}|}{|a_0|^{n-m+2}} + \cdots + \frac{|a_n - a_{n-1}|}{|a_0|^{n-1}} + \frac{|a_n - a_n|}{|a_0|} \}$$

$$\geq |a_0||z|^{n+1} \{ |z| - \frac{1}{|a_0|}\{|a_n| - |a_0| + T_1\} \}$$

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk
$$|z| \leq \frac{1}{|a_0|}\{|a_n| - |a_0| + T_1\}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in
$$|z| \leq \frac{1}{|a_0|}\{|a_n| - |a_0| + T_1\}$$

Since $P(z) = z^n J_{-\frac{1}{z}}$ it followed by replacing $z$ by $\frac{1}{z}$, all the zeros of $P(z)$ lie in
$$|z| \geq \frac{|a_0|}{|a_n| - |a_0| + T_1},$$

if both $n$ and $(n-m)$ are even or odd.

Hence all the zeros of $P(z)$ does not vanish in the disk
$$|z| < \frac{|a_0|}{|a_n| - |a_0| + T_1}$$
if both \( n \) and \( (n-m) \) are even or odd

where

\[
T_1 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right)
\]

Similarly, we can also prove that if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even degree polynomials. For this, we can rearrange the terms of the given polynomial and compute as above. That is all the zeros of \( P(z) \) do not vanish in the disk.

\[
|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_2}
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

where

\[
T_2 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \right)
\]

This completes the proof of Theorem 2.

**Proof of Theorem 3.**

Let

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0
\]

be a polynomial of degree \( n \).

Let us consider the polynomial \( J(z) = z^n P\left(\frac{1}{z}\right) \)

and \( R(z) = (z - 1)J(z) \) so that

\[
R(z) = (z - 1)\left( a_n z^n + a_{n-1} z^{n-1} + \cdots + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \right)
= a_0 z^{n+1} - \left( (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{n-1} - a_n) z + a_n \right)
\]

Also if \( |z| > 1 \) then \( \frac{1}{|z|^{n+k}} < f \) for \( i = 0,1,2, \ldots, n-1 \).

Now

\[
|R(z)| \geq \left| a_0 \right| |z|^{n+1} - \left| a_0 - a_1 \right| |z|^n + \left| a_1 - a_2 \right| |z|^{n-1} + \cdots + \left| a_{n-1} - a_n \right| |z| + \left| a_n \right|
\]

\[
\geq \left| a_0 \right| |z|^n \left[ |z| - \frac{1}{|z|} \left( \left| a_0 - a_1 \right| + \left| a_1 - a_2 \right| + \cdots + \left| a_{n-1} - a_n \right| \right) \right]
\]

\[
\geq \left| a_0 \right| |z|^n \left[ |z| - \frac{1}{|z|} \left( \left| a_0 \right| + \left| a_1 \right| + \left| a_2 \right| + \cdots + \left| a_{n-2} \right| + \left| a_{n-1} \right| + \left| a_n \right| \right) \right]
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

where \( U_1 = 2\left( (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right) \)

\[
\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \left( a_0 + |a_n| + a_n + U_1 \right)
\]

This shows that all the zeros of \( R(z) \) whose modulus is greater than 1 lie in the closed disk

\[
|z| \leq \frac{1}{|a_0|} \left( a_0 + |a_n| + a_n + U_1 \right)
\]

But those zeros of \( R(z) \) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of \( R(z) \) and hence \( J(z) \) lie in

\[
|z| \leq \frac{1}{|a_0|} \left( a_0 + |a_n| + a_n + U_1 \right)
\]

Since \( P(z) = z^n J\left(\frac{1}{z}\right) \), it followed by replacing \( z \) by \( \frac{1}{z} \).

all the zeros of \( P(z) \) lie in

\[
|z| \geq \frac{|a_0|}{a_0 + |a_n| + a_n + U_1}
\]

if both \( n \) and \( (n-m) \) are even or odd.

Hence all the zeros \( P(z) \) do not vanish in the disk

\[
|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_2}
\]

if both \( n \) and \( (n-m) \) are even or odd

where \( U_2 = 2\left( (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right) \)

Similarly, we can also prove that if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even degree polynomials. For this, we can rearrange the terms of the given polynomial and compute as above. That is all the zeros \( P(z) \) do not vanish in the disk.

\[
|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_2}
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

where

\[
U_2 = 2\left( (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right)
\]

This completes the proof of Theorem 3.

**Proof of the Theorem 4.**

Let

\[
P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0
\]

be a polynomial of degree \( n \).
Let us consider the polynomial \( f(z) = z^n P(z) \)

and \( R(z) = (z - 1)f(z) \) so that

\[
R(z) = (z - 1) \left( a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{m-1} + a_m \right) + a_{m+1} z^{m+1} + \cdots + a_{n-1} z + a_n
\]

Also if \( |z| > 1 \) then \( \frac{1}{|z|^{n-i}} < f \) for \( i = 0, 1, 2, \ldots, n - 1 \).

Now

\[
|R(z)| \geq |a_0||z|^{n+1} - \left\{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m+2} + \cdots + |a_{n-1} - a_n||z| + |a_n| \right\}
\]

\[
\geq |a_0||z|^{n}
\]

\[
|z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right\}
\]

\[
\geq |a_0||z|^{n}
\]

\[
|z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right\}
\]

if both \( n \) and \( n-m \) are even or odd.

Hence all the zeros \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{|a_0| + a_n - a_0 + V_1}
\]

Similarly we can also prove for if \( n \) is even and \( n-m \) is odd (or) if \( n \) is odd and \( n-m \) is even.

This completes the proof of the Theorem 4.

References

[1] G. Eneström, Remarques sur un théorème relatif aux racines de l’équation \( a_n + \cdots a_0 = 0 \) où tous les coefficients sont positifs, Tôhoku Math. J. 18 (1920), 34-36.


