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# Zero-Free Region for Polynomials with Restricted Coefficients

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**Abstract:** *In this paper we prove some extension of the Eneström-Kakeya theorem says that. Let P(z)=* $\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree nsuch that* $0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_n$ *then all the zeros of P(z) lie in |z|≤1. By relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.*

**Mathematics Subject Classification:** *30C10, 30C15.*

**Keywords:** Zeros of polynomial, Eneström- Kakeya theorem

## 1. Introduction

The well known Results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following.

**Theorem ($A_1$).** LetP(z)=$\sum_{i=0}^{n} a_i z^i$*be a polynomial of degree nsuch that* $0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_n$ then allthe zeros of P(z) lie in |z|≤1.

Applying the above result to the polynomial $z^n$P($\frac{1}{z}$) we get the following result:

**Theorem ($A_2$).***If P(z) =*$\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that*
$0 < a_n \le a_{n-1} \le a_{n-2} \le \cdots \le a_0$*then P(z) does not vanish in* |z|<1

In the literature [3-10], there exist several extensions and generalizations of the Eneström-Kakeya Theorem.
In this paper we give generalizations of the above mentioned results. In fact we prove the following results:

**Theorem 1.** Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$ and 0≤ $m < n$ with real coefficients such that

$$a_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \ge a_{n-m-1} \le a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n$$

if both n and (n-m) are even or odd
(OR)

$$a_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \le a_{n-m-1} \ge a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0+|a_n|-a_n+S_1}$
if both n and (n-m) are even or odd

where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0+|a_n|-a_n+S_2}$
if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

**Corollary 1.**Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$ and 0≤ $m < n$ with positive real coefficients such that

$$a_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \ge a_{n-m-1} \le a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n$$

if both n and (n-m) are even or odd
(OR)

$$a_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \le a_{n-m-1} \ge a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{a_0+S_1}$
if both n and (n-m) are even or odd

where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{a_0+S_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

**Remark 1.** By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n$,in theorem 1, then it reduces to Corollary 1.

**Theorem 2.** Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and 0$\leq m < n$ with real coefficients such that

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_1}$

if both n and (n-m) are even or odd

where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $||z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

**Corollary 2.**Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and 0$\leq m < n$ with positive real coefficients such that

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{T_1 - a_0}$

if both n and (n-m) are even or odd

where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $||z| < \frac{a_0}{T_2 - a_0}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

**Remark 2.** By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n.$ in theorem 2, then it reduces to Corollary 4.

**Theorem 3.** Let LetP(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and 0$\leq m < n$ with real coefficients such that

$$a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_1}$

if both n and (n-m) are even or odd

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$

**Corollary 3.**Let LetP(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and 0$\leq m < n$ with positive realcoefficients such that

$$a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{a_0 + 2a_n + U_1}$

if both n and (n-m) are even or odd

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{a_0 + 2a_n + U_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$.

**Remark 3.** By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n$ in theorem 3, then it reduces to Corollary 3.

**Theorem 4.**Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and $0 \leq m < n$ with real coefficients such that

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n| + a_n - a_0 + V_1}$

if both n and (n-m) are even or odd

where $V_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $||z| < \frac{|a_0|}{|a_n| + a_n - a_0 + V_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

**Corollary 4.**Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and $0 \leq m < n$ with positive real coefficients such that

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if both n and (n-m) are even or odd

(OR)

$$a_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then (i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{2a_n - a_0 + V_1}$

if both n and (n-m) are even or odd

where $V_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m})\,]$

(ii) ) all the zeros of P(z) does not vanish in the disk $||z| < \frac{a_0}{2a_n - a_0 + V_2}$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

**Remark 4.**By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n,$ in theorem 4, then it reduces to Corollary 4.

## 2. Proofs of the Theorems

**Proof of the Theorem 1.**

Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n \mathrm{P}(\frac{1}{z})$

and $R(z) = (z-1)J(z)$so that

$$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)$$

$$= a_0 z^{n+1} - \{\,(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\,\}$$

Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for\ i = 0,1,2,\ldots,n-1.$

Now $$|R(z)| \geq |a_0||z|^{n+1} - \{\ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$$

$$\geq |a_0||z|^n [\,|\mathrm{z}| - \frac{1}{|a_0|}\{\,|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$$

$$\geq |a_0||z|^n \quad [|\mathrm{z}| \quad - \frac{1}{|a_0|}\{|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |\,a_n|\,\}]$$

$$\geq |a_0||z|^n [\,|\mathrm{z}| - \frac{1}{|a_0|}\{(\,a_0 - a_1) + (\,a_2 - a_1) + (\,a_2 - a_3) + \cdots + (\,a_{n-m} - a_{n-m-1}) + (\,a_{n-m} - an-m+1+\ldots+ \quad an-3-an-2+ \quad an-2-an-1+ an-1-an+|an|\,\}]$$

if both n and (n-m) are even or odd

$$= |a_0||z|^n [\,|\mathrm{z}| - \frac{1}{|a_0|}\{a_0 + |a_n| - a_n + S_1\}]$$

where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

$$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{a_0 + |a_n| - a_n + S_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{a_0 + |a_n| - a_n + S_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk.Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{a_0 + |a_n| - a_n + S_1\}$$

Since P(z) = $z^n$J($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{a_0 + |a_n| - a_n + S_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + S_1}$$

if both n and (n-m) are even or odd
where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degreepolynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zerosP(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + S_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

This completes the proof of the Theorem 1.

**Proof of the Theorem 2.**

Let P(z)= $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n$P($\frac{1}{z}$)

and $R(z) = (z-1)J(z)$ so that

$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n)$

$= a_0 z^{n+1} - \{\ (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots + (a_{n-1} - a_n)z + a_n\}$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for\ i = 0,1,2,\ldots,n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{\ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\}$

$\geq |a_0||z|^n[\,|\mathrm{z}| - \frac{1}{|a_0|}\{\,|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$

$\geq |a_0||z|^n\ [|\mathrm{z}| - \frac{1}{|a_0|}\{|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |\,a_n|\,\}]$

$\geq |a_0||z|^n[\,|\mathrm{z}| - \frac{1}{|a_0|}\{(\,a_1 - a_0) + (\,a_1 - a_2) + (\,a_3 - a_2) + \cdots + (\,a_{n-m-1} - a_{n-m}) + (\,a_{n-m} -$ $an-m+1+\ldots+ \quad an-3-an-2+ \quad an-2-an-1+ an-1-an+|an|\,\}]$
if both n and (n-m) are even or odd

$$= |a_0||z|^n[\,|\mathrm{z}| - \frac{1}{|a_0|}\{|a_n| - a_n - a_0 + T_1\}\,]$$

where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

$$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{|a_n| - a_n - a_0 + T_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{|a_n| - a_n - a_0 + T_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{|a_n| - a_n - a_0 + T_1\}$$

Since P(z) = $z^n$J($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_n| - a_n - a_0 + T_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_1}$$

if both n and (n-m) are even or odd
where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degreepolynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2\big[(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m})$- $(a_2 +$ *a4+…+ an−m−3+an−m−1.*

This completes the proof of the Theorem 2.

**Proof of the Theorem 3.**

Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n \mathrm{P}(\frac{1}{z})$

and $R(z) = (z-1)J(z)$ so that

$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)$

$= a_0 z^{n+1} - \{\ (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\}$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for\ i = 0,1,2,\dots,n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{\ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$

$\geq |a_0||z|^n[\ |\mathrm{z}| - \frac{1}{|a_0|}\{\,|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$

$\geq |a_0||z|^n \quad [|\mathrm{z}| \quad - \frac{1}{|a_0|}\{|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |\,a_n|\,\}]$

$\geq |a_0||z|^n[\ |\mathrm{z}| - \frac{1}{|a_0|}\{(\,a_0 - a_1) + (\,a_2 - a_1) + (\,a_2 - a_3) + \cdots + (\,a_{n-m} - a_{n-m-1}) + (\,a_{n-m+1} -$ *an−m+…+ an−2−an−3+ an−1−an−2+ an−an−1+|an| }]*
if both n and (n-m) are even or odd

$= |a_0||z|^n[\ |\mathrm{z}| - \frac{1}{|a_0|}\{a_0 + |a_n| + a_n + U_1\}]$

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{a_0 + |a_n| + a_n + U_1\}$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{a_0 + |a_n| + a_n + U_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{a_0 + |a_n| + a_n + U_1\}$$

Since P(z) = $z^n \mathrm{J}(\frac{1}{z})$ it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{a_0 + |a_n| + a_n + U_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_1}$$

if both n and (n-m) are even or odd
where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degreepolynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{a_0 + |a_n| + a_n + U_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$.

This completes the proof of the Theorem 3.

**Proof of the Theorem 4.**

Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n \mathrm{P}(\frac{1}{z})$

and $R(z) = (z-1)J(z)$ so that

$$R(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)$$

$$= a_0 z^{n+1} - \{ (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n \}$$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|z|^{n-i}} < for\ i = 0,1,2,\ldots,n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}$

$$\geq |a_0||z|^n [\, |\mathrm{z}| - \frac{1}{|a_0|}\{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^2} + \frac{|a_3 - a_4|}{|z|^3} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{m-1}} + \frac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \}]$$

$$\geq |a_0||z|^n \quad [|\mathrm{z}| \quad - \frac{1}{|a_0|}\{|a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |a_n| \}]$$

$\geq |a_0||z|^n [\, |\mathrm{z}| - \frac{1}{|a_0|}\{( a_1 - a_0) + ( a_1 - a_2) + ( a_3 - a_2) + \cdots + ( a_{n-m-1} - a_{n-m}) + ( a_{n-m+1} -$ *an−m+…+ an−2−an−3+ an−1−an−2+ an−an−1+|an|* }]
if both n and (n-m) are even or odd

$$= |a_0||z|^n [\, |\mathrm{z}| - \frac{1}{|a_0|}\{|a_n| + a_n - a_0 + V_1\} ]$$

where $V_1 = 2[\, (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) \,]$

$$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{|a_n| + a_n - a_0 + V_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{|a_n| + a_n - a_0 + V_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{|a_n| + a_n - a_0 + V_1\}$$

Since P(z) = $z^n \mathrm{J}(\frac{1}{z})$ it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_n| + a_n - a_0 + V_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0 + V_1}$$

if both n and (n-m) are even or odd
where $V_1 = 2[\, (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) \,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degreepolynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_n| + a_n - a_0 + V_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\, (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \,]$.

This completes the proof of the Theorem 4.

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