Stochastic Epidemic Model with Poisson Infection Rate without Removal

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Abstract: ***In this paper, we have developed a stochastic epidemic model with Poisson infection rate without removal. Poisson infection rate is depending on the number of persons infected in the system.***

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1. Introduction

The occurrence of more cases of a disease than would be expected in a community or region during a given time period is called an epidemic. An epidemic is the rapid spread of infectious disease to a large number of people in a given population within a short period of time, usually two weeks or less. An epidemic may be restricted to one location; however, if it spreads to other countries or continents and affects a substantial number of people, it may be termed a pandemic **[8]**.

In mathematics and physics, a deterministic system is a system in which no randomness is involved in the development of future states of the system **[9]**. A deterministic model will thus always produce the same output from a given starting condition or initial state **[10]**. For example, physical laws that are described by differential equations represent deterministic systems, even though the state of the system at a given point in time may be difficult to describe explicitly. Markov chains and other random walks are not deterministic systems, because their development depends on random choices.

Stochastic means being or having a random variable. A stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more inputs over time. The random variation is usually based on fluctuations observed in historical data for a selected period using standard time series techniques. Distributions of potential outcomes are derived from a large number of simulations which reflect the random variation in the input.

In probability theory, a Poisson process is a stochastic process that counts the number of events and the time points at which these events occur in a given time interval. The time between each pair of consecutive events has an exponential distribution with parameter λ and each of these inter-arrival times is assumed to be independent of other inter-arrival times. The process is named after the Poisson distribution introduced by French mathematician Simeon Denis Poisson **[5]**. It describes the time of events in radioactive decay **[2]**, telephone calls **[7]** or request for documents on a web server under certain conditions **[1]**, and many other phenomena, where events occur independently from each other. The number of arrivals $N(t)$ in a finite interval of length t obeys the Poisson (λt) distribution, $P\{N(t)=n\}=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$. Moreover, the numbers of arrivals $N(t_1,t_2)$ and $N(t_3,t_4)$ in non-overlapping intervals $(t_1 \le t_2 \le t_3 \le t_4)$ are independent.

An infection rate is an estimate of the rate of progress of a disease, based on proportional measures of the extent of infection at different times.

In probability theory, the probability generating function of a discrete random variable is a power series representation of the probability mass function of the random variable. Probability generating functions are often employed for their succinct description of the sequence of probabilities $\Pr(X=i)$ in the probability mass function for a random variable X, and to make available the well developed theory of power series with non-negative coefficients.

Definition 1.1 [4]

Consider a random variable X, i.e, a discrete random variable taking non-negative values.

Write $P_k = P(X=k), \qquad k=0,1,2,...$

The probability generating function of X is defined as

$$G_X(S)=\sum_{k=0}^{\infty} P_k S^k = E(S^X).$$

Note that $G_X(1)=1,$ so the series converges absolutely for $|S|\le 1.$ Also $G_X(0)=P_0.$

Theorem 1.2 [4] (Total and Compound Probability)
Let $A_1, A_2, ..., A_n$ be a partition of Ω. For any event *B*,

$$\Pr(B) = \sum_{j=1}^{n} \Pr(A_j)\Pr(B/A_j).$$

In this paper, we have developed a stochastic epidemic model with Poisson infection rate without removal. Poisson infection rate is depending on the number of persons infected in the system.

2. Epidemic Model

In this section we study the characterizations of the stochastic epidemic model with Poisson infection rate without removal.

2.1. Stochastic Epidemic Model with Poisson Infection Rate without Removal

Let $P_n(t)$ be the probability that there are *n* susceptible persons in the system and let *m* be the number of infected persons in the system, $f_j(m)\Delta t + O(\Delta t)$ give the probability that the number changes to $n+j$ in the time interval $(t, t+\Delta t)$. Here j is any positive integer. **[3]**

Since, Δt is very small and $O(\Delta t)$ denotes an infinitesimal which is such that

$$\frac{O(\Delta t)}{\Delta t} \to 0 \quad as\ \Delta t \to 0. \qquad \textbf{(1)}$$

So that if $j > 1$, $f_j(m) = 0$.

In this case, let us assume that,

$$f_j(m) = \begin{cases} \dfrac{e^{-\lambda}\lambda^m}{m!}, & m \ge 0,\ j = 1 \\ 0, & \text{otherwise} \end{cases} \qquad \textbf{(2)}$$

The probability that there is no change in the time interval $(t, t+\Delta t)$ is then given by

$$1 - \left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t) \qquad \textbf{(3)}$$

Using the theorems of total and compound probabilities **[3, 6]**, we get

$P_n(t+\Delta t) =$ Probability of *n* susceptible at time *t* and no susceptible at time Δt + (*n*-1) susceptible at time *t* and 1 susceptible at time Δt.

$$= P_n(t)\left(1 - \left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t\right) + P_{n-1}(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t) \qquad \textbf{(4)}$$

So that,

$$P_n(t+\Delta t) = P_n(t) - P_n(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + P_{n-1}(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t)$$

$$P_n(t+\Delta t) - P_n(t) = -P_n(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + P_{n-1}(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -P_n(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right) + P_{n-1}(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right) + \frac{O(\Delta t)}{\Delta t}, \qquad n \ge 1 \qquad \textbf{(5)}$$

Proceeding to the limit as $\Delta t \to 0$, we obtain

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -P_n(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right) + P_{n-1}(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right) + 0 \quad \text{(By using \textbf{(1)})}$$

$$\frac{d}{dt}P_n(t) = -\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[P_n(t) - P_{n-1}(t)\right], \qquad n \ge 1. \qquad \textbf{(6)}$$

For the case when *n* = 0,

$P_0(t+\Delta t) = P_0(t) \times$ There is no susceptible in Δt.

$$= P_0(t)\left(1 - \left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t)\right)$$

$$= P_0(t) - P_0(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t)$$

$$P_0(t+\Delta t) - P_0(t) = -P_0(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\Delta t + O(\Delta t).$$

Dividing on both sides by Δt and taking the limit as $\Delta t \to 0$, we have

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -P_0(t)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right) + 0 \quad \text{(By using \textbf{(1)})}$$

$$\frac{d}{dt}P_0(t) = -\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)P_0(t) \qquad \textbf{(7)}$$

Multiplying **(6)** and **(7)** by x^n, summing for all *n* and using the definition of the probability generating function **[6]**, namely

$$\varphi(x,t) = \sum_{n=0}^{\infty} P_n(t)x^n, \qquad \textbf{(8)}$$

We get,

$$\frac{d}{dt}\sum_{n=0}^{\infty}P_n(t)x^n=\frac{d}{dt}\left[P_0(t)+\sum_{n=1}^{\infty}P_n(t)x^n\right]$$

$$\frac{d}{dt}\varphi(x,t)=\frac{d}{dt}P_0(t)+\left(\sum_{n=1}^{\infty}\frac{d}{dt}P_n(t)x^n\right). \quad \textbf{(9)}$$

By using the result **(6)**, we get

$$\frac{d}{dt}\varphi(x,t)=\frac{d}{dt}P_0(t)-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\sum_{n=1}^{\infty}\left[P_n(t)-P_{n-1}(t)\right]x^n. \quad \textbf{(10)}$$

Now, the definition of probability generating function, we have

$$\varphi(x,0)=\sum_{n=0}^{\infty}P_n(0)x^n$$

$$=P_0(0)+P_1(0)x+P_2(0)x^2+\ldots$$

$$\varphi(x,0)=1; \quad \textbf{(11)}$$

$$\sum_{n=1}^{\infty}P_n(t)x^n=\varphi(x,t)-P_0(t); \quad \textbf{(12)}$$

And $$\sum_{n=1}^{\infty}P_{n-1}(t)x^n=x\varphi(x,t) \quad \textbf{(13)}$$

Using the results **(7)**, **(12)** and **(13)** in **(10)**, we get

$$\frac{d}{dt}\varphi(x,t)=-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)P_0(t)-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[\varphi(x,t)-P_0(t)-x\varphi(x,t)\right]$$

$$=-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[P_0(t)+\varphi(x,t)-P_0(t)-x\varphi(x,t)\right]$$

$$=-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[\varphi(x,t)-x\varphi(x,t)\right]$$

$$=-\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[(1-x)\varphi(x,t)\right]$$

$$=\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\left[(x-1)\varphi(x,t)\right]$$

$$\frac{d\varphi(x,t)}{\varphi(x,t)}=\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)dt \quad \textbf{(14)}$$

Integrating **(14)** on both sides, we get

$$\log(\varphi(x,t))=\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t+\log C$$

$$\log(\varphi(x,t))-\log C=\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t$$

$$\log\left(\frac{\varphi(x,t)}{C}\right)=\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t$$

Taking exponential on both sides, we get

$$\frac{\varphi(x,t)}{C}=\exp\left(\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t\right)$$

$$\varphi(x,t)=C\exp\left(\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t\right) \quad \textbf{(15)}$$

Let t = 0 in **(15)**, we get

$$\varphi(x,0)=C$$

By using the result **(11)**, we get C = 1.
Therefore, the result **(15)** can be written as

$$\varphi(x,t)=\exp\left(\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)(x-1)t\right)$$

$$=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\exp\left(\frac{e^{-\lambda}\lambda^m}{m!}xt\right) \quad \textbf{(16)}$$

Hence, the probability generating function of the given system is given by

$$\varphi(x,t)=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\sum_{n=0}^{\infty}\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^n\frac{(xt)^n}{n!}$$

So that,

$P_n(t)\equiv$ Co-efficient of x^n in $\varphi(x,t)$.

$$=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^n\frac{t^n}{n!}, \qquad m,n\geq 0. \quad \textbf{(17)}$$

Remark 2.2

$P_n(t)$ is a probability function.

Proof

If the probability $P_n(t)$ is a probability function then

$$\sum_{n=0}^{\infty}P_n(t)=1.$$

Let

$$P_n(t)=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^n\frac{t^n}{n!}, \qquad m,n\geq 0.$$

Now, $$\sum_{n=0}^{\infty}P_n(t)=\sum_{n=0}^{\infty}\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^n\frac{t^n}{n!}$$

$$=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\sum_{n=0}^{\infty}\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^n\frac{t^n}{n!}$$

$$=\exp\left(-\frac{e^{-\lambda}\lambda^m}{m!}t\right)\left(1+\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)\frac{t}{1!}+\left(\frac{e^{-\lambda}\lambda^m}{m!}\right)^2\frac{t^2}{2!}+\ldots\right)$$

.

Put $X=\frac{e^{-\lambda}\lambda^m}{m!}t$, we get

$$\sum_{n=0}^{\infty}P_n(t)=\exp(-X)\left(1+\frac{X}{1!}+\frac{X^2}{2!}+\ldots\right)$$

$$= e^{-X}\, e^{X}$$

$$\sum_{n=0}^{\infty} P_n(t) = 1.$$

Hence, the given probability $P_n(t)$ is a probability function.

Example 2.3

There are 5000 people living in a town. Initially it was found that an epidemic occurred and 1% of the people were infected. Also, it was measured that the infection rate is 2.4.

Now, the Poisson infection rate is $\lambda = 0.2177$.
Hence, the probability of n infections after 4 units of time is

$$P_n(t) = \frac{e^{-4\lambda}(4\lambda)^n}{n!}.$$

For $n = 1$, $P_1(t) = 0.3645.$

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