

2. Basic Definitions and Concepts

2.1 Reliability

Reliability is defined as the probability of a device (or an item) performing its purpose for the period intended under the given operating conditions. It can also be defined as the probability of non-failure. If $F(t)$ is the failure probability, then $1-F(t)$ gives the non-failure probability. Thus, the Reliability of the device (or an item) for time $T = t$ (ie., the device functions satisfactorily for time $T \geq t$ is

$$R(t) = 1 - F(t)$$

$$= 1 - \int_{-\infty}^t f(x) dx$$

$$\text{or } R(t) = \int_t^{\infty} f(x) dx$$

where, $R(t)$ is the Reliability at time t

2.2 Series System

In a Series System, all Components in the System should be operating to maintain the required operation of the System. Thus, the failure of any one Component of the System will cause failure of the whole System.

Series configuration

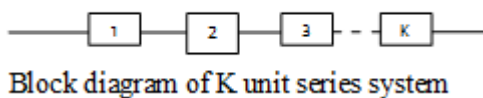


Fig. 1
The System Reliability Estimate of the Series System is given by

$$\hat{R}_s = \prod_{i=1}^m P(X_i)$$

where $P(X_i)$ is the Reliability of the i^{th} component

2.3 Parallel System

In a Parallel System, the System operates if one or more Components operate, and the System fails if all the Components fail. The Parallel n -components are represented by the following block diagram.

Parallel configuration

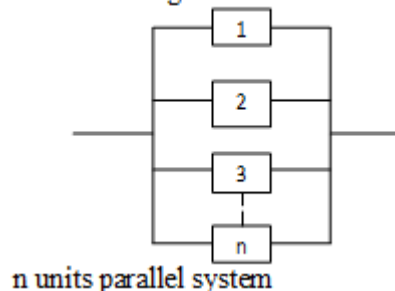


Fig. 2

$$R_p(t) = 1 - \prod_{i=1}^n (1 - P(X_i))$$

where $P(X_i)$ is the probability of the i^{th} component.

2.4 Redundancy

We can improve the System Reliability by the technique of introducing Redundancies. This involves the deliberate creation of the new Parallel paths in the System. We have observed that if two elements with probabilities of success $P(a)$ and $P(b)$ are connected in Parallel, the probability $P(a \text{ or } b)$ is

$$P(a \text{ or } b) = P(a) + P(b) - P(a \text{ and } b)$$

$$= P(a) + P(b) - P(a) \times P(b)$$

Assuming that the elements are independent, since both $P(a)$ and $P(b)$ are individually less than one, their product is always less than $P(a)$ or $P(b)$. Hence, $P(a \text{ or } b)$ is always greater than either $P(a)$ or $P(b)$. Although either one of the elements is sufficient for the successful operation of the system, we deliberately use both elements so as to increase the Probability of Success, thus causing the System to become Redundant.

2.5 Fault tree Analysis (FTA) approach

FTA is a top-down approach of a system analysis that is used to determine the possible occurrence of undesirable events or failures. Over the years, the method has gained favor over other reliability analysis approaches because of its versatility in degree of detail of complex systems. There are many symbols used to construct fault trees. The basic four symbols are

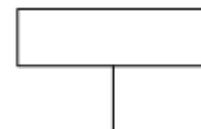


Figure 3

It denotes a fault event that occurs from the logical combination of fault events through the input of logic gates such as OR and AND



Figure 4

It denotes a basic fault event

Output event (Faults)
OR gate



Input event (faults)

Figure 5

It denotes the output fault event if one or more of input fault events occur.

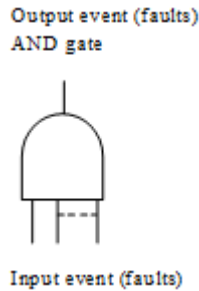


Figure 6

It denotes that an output fault tree event occurs if all the input fault events occur

3. General Series Parallel System

The System consists of stage 1, stage 2, ..., stage k connected in Series. Each stage contains a number of Redundant elements,

Stage i consisting of n_i Redundant elements connected in Parallel. The Reliability of the System is the product of the Reliabilities of each stage. Stage i with n_i will have the Reliability, $R_i = 1 - [1 - P(X_{i1})][1 - P(X_{i2})] \dots [1 - P(X_{in_i})]$

$$= 1 - \prod_{j=1}^{n_i} [1 - P(X_{ij})]$$

Therefore, the System Reliability is

$$R(s) = R_1 R_2 R_3 \dots R_k$$

$$= \prod_{i=1}^k \left\{ 1 - \prod_{j=1}^{n_i} [1 - P(X_{ij})] \right\} \dots \dots \dots (1)$$

Now, for Example, Consider a system which has 2 subsystems connected in series (say X_1 and X_2) each system has 2 components connected in parallel (say X_{11} and X_{12}). The Fault tree analysis diagram of the system is given below,

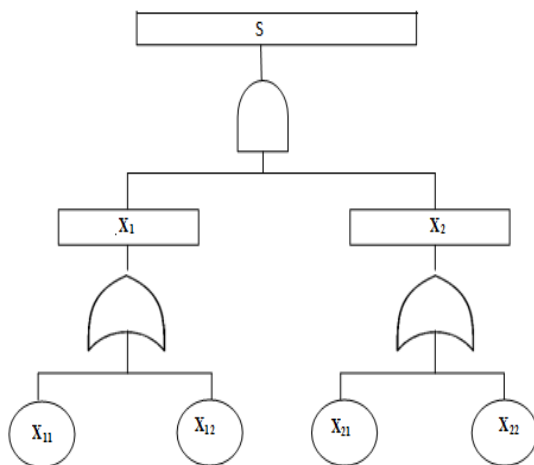


Figure 7

Dynamic Programming Formulation to a Reliability Model

Reliability is a most important requirement for many Medical Systems, such as those designed for multistage

operation systems. Such Systems can be considered as a series of "black boxes" or subsystems. If by chance a subsystem fails, the entire System fails. To avoid chance of failure, that is to say, increase the Reliability of the whole System, Redundancy in some or all of the Systems is incorporated. This means that instead of using just one Subsystems in stage j, two or more are connected in parallel along with switching circuits in such a way that if one operating Subsystem fails, one of the Redundant Subsystem is automatically switched on.

Notations

- c_j - Cost of one subsystem in stage j.
 - M - Total Money available for the entire system.
 - $1 + x_j$ - The number of subsystems in stage j.
 - $p_j(x_j)$ - The probability that the jth stage operates successfully.
 - x_j - is the redundancy in the stage
- where $x_j \geq 0, x_j \leq m_j$ some given number.

Assuming that $p_j(x_j)$ is independent of what is done for other stages, the reliability of the whole system is $p_1(x_1) \times p_2(x_2) \times \dots \times p_n(x_n)$. The total cost involved is

$$\sum_{j=1}^n c_j (1 + x_j).$$

Application To a Reliability Model

x_j	$p_1(x_1)$	$p_2(x_2)$	$p_3(x_3)$
0	0.6	0.8	0.7
1	0.7	0.8	0.8
2	0.9	0.9	0.9

Hence for Maximum Reliability, we have the problem Maximise:

$$z = p_1(x_1) \times p_2(x_2) \times \dots \times p_n(x_n) \dots \dots \dots (2)$$

$$\text{s.t. } \sum_{j=1}^n c_j x_j \leq M - \sum_{j=1}^n c_j = b \text{ (say)}$$

$$\text{Integer } x_j \geq 0 \text{ and } \leq m_j$$

The optimal values in the 'n' stages are given by the recursive relations

$$\xi_{k-1} = \xi_k - c_k x_k, (k=n, n-1, \dots, 2) \text{ with } \xi_n = b \dots \dots \dots (3)$$

Now,

$$F_k(\xi_k) = \max_{0 \leq x_k \leq \min\{m_k, \lfloor \xi_k / c_k \rfloor\}} \{f_k(x_k) \times F_{k-1}(\xi_k - c_k x_k)\}$$

For $k = n, n-1, \dots, 2$ and

$$F_1(\xi_1) = \max_{0 \leq x_1 \leq \min\{m_1, \lfloor \xi_1 / c_1 \rfloor\}} \{f_1(x_1)\}, \xi_1 = 0, 1, 2, \dots, b.$$

The Maximum Reliability is $F_n(\xi_n)$. Recursion is started and carried through the stages 2, 3, ..., n.

4. A Medical Example

A Health Organization conducts a free treatment scheme for a System consisting of three category of patients (regarded as subsystems) infected by a particular disease. The three patients are under three different stages of the disease (less affected, moderately affected, more affected). Three Doctors (Redundancies of the subsystems) are available for treatment of these patients with per unit cost of the three categories of patients as 1, 3 and 2 respectively. The probability of the subsystems are given below

	$f_1(x_1)$			
x_1	0	1	2	$F_1(\xi_1)$
ξ_1				
0	0.6			0.6
1	0.6	0.7		0.7
≥ 2	0.6	0.7	0.9	0.9

If the total allocation cannot exceed Rs.1, 30,000, We determine the redundancies for the most reliable system. The mathematical model is

$$\begin{aligned} \text{Maximize: } z &= p_1(x_1) \times p_2(x_2) \times p_3(x_3) \\ \text{s.t } 1x_1 + 3x_2 + 2x_3 &= 13 - (1 + 3 + 2) = 7 \\ \text{Integer } x_j &\geq 0 \text{ and } \leq 2 \end{aligned}$$

From equation (1) we know that $\xi_n = b$

Choosing $n=3$ we have $\xi_3 = 7$

The recurrence relations in Equation (1) are

$$\begin{aligned} \xi_2 &= 7 - 2x_3 \\ \xi_1 &= 7 - 3x_2 - 2x_3 \\ &= \xi_2 - 3x_2 \end{aligned}$$

The Admissible values of ξ_2 and ξ_1 are

x_3	0	1	2
ξ_2	7	5	3

and

x_2	0	0	0	1	1	1	2
x_3	0	1	2	0	1	2	0
ξ_1	7	5	3	4	2	0	1

At the first stage the optimal value is given by

$$F_1(\xi_1) = \max_{0 \leq x_1 \leq \min\{2, [\xi_1/1]\}} \{f_1(x_1)\}, \xi_1 = 0, 1, 2, 3, 4, 5, 7. \quad \dots\dots\dots (4)$$

Now take $\xi_1=0$ and $x_1=0, 1, 2$

$$F_1(0) = \max_{0 \leq x_1 \leq \min\{2, 0\}} \{f_1(x_1)\} = \max\{0.6\} = 0.6$$

At $\xi_1=1$ and $x_1=0, 1, 2$ we have

$$F_1(1) = \max_{0 \leq x_1 \leq \min\{2, 1\}} \{f_1(x_1)\} = \max\{0.6, 0.7\} = 0.7$$

At $\xi_1=2$ and $x_1=0, 1, 2$ we have

$$F_1(2) = \max_{0 \leq x_1 \leq \min\{2, 2\}} \{f_1(x_1)\} = \max\{0.6, 0.7, 0.9\} = 0.9$$

The table below shows the first stage values.

Similarly, at the second Stage

$$F_2(\xi_2) = \max_{0 \leq x_2 \leq \min\{2, [\xi_2/2]\}} \{f_2(x_2) \times F_1(\xi_1)\}, \xi_2 = 3, 5, 7.$$

The table for this is given below

	$f_2(x_2) \times F_1(\xi_2 - 3x_2)$			
x_2	0	1	2	$F_2(\xi_2)$
ξ_2				
3	0.8 x 0.9	0.8 x 0.6		0.72
5	0.8 x 0.9	0.8 x 0.7		0.72
7	0.8 x 0.9	0.8 x 0.9	0.9 x 0.7	0.72

At the final Stage

$$F_3(\xi_3) = \max_{0 \leq x_3 \leq \min\{2, [\xi_3/2]\}} \{f_2(x_2) \times F_1(\xi_1)\}, \xi_3 = 7.$$

For which we have the table

	$f_3(x_3) \times F_2(7 - 2x_3)$			
x_3	0	1	2	$F_3(\xi_3)$
ξ_3				
7	0.7x0.72	0.8x0.72	0.9 x 0.72	0.648

From the table, we get the Maximum is for $x_3 = 2$ and by trace back $x_2 = 0$, $x_1 = 2$. The maximum Reliability is 0.648 at a cost of $3 \times 1 + 1 \times 3 + 3 \times 2 = 12$ (in tens of lakhs of rupees).

But from the equation (1) Reliability of the System for the given numerical example is determined as 0.9633. This clearly shows that when the Cost of the System is taken into consideration, we find that the Reliability of the System is reduced by 31 percent.

5. Conclusion

Dynamic programming is a newly developed Mathematical technique which is often useful for making a sequence of inter-related decisions. We have made use of this technique to allocate the minimum resource for a System and find its Reliability. Comparative study tells us that the System Reliability decreases when Cost of the System is taken care

off, where we have considered the Cost of the System not to exceed the Total money allocated for the entire System.

6. Acknowledgement

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