Application of Dynamic Programming Technique to Reliability Model in Medical Field

M. Reni Sagayaraj[1], A. Merceline Anita[2], A. Chandra Babu[3], S. Gowtham Prakash[4]

[1, 2, 4]Department of Mathematics, Sacred Heart College, Vellore Dt.

[3]Department of Mathematics, Noorul Islam University, Kanyakumari Dt.

Abstract: ***In this Paper, we make a study on how to apply Dynamic Programming Technique to Reliability model, which is applicable to any System that can be modeled into a Series-Parallel configuration. We determine the Reliability of the System using Fault Tree Analysis Approach and also, when the cost of the System is taken into consideration, then the Reliability of the System is determined using Dynamic Programming.***

Keywords: Reliability, Series- Parallel Configuration, Redundancy, Dynamic Programming, Fault Tree Analysis.

1. Introduction

Reliability R (t), is the probability that a device or an item performs its function adequately over the time interval (0,t). The device under consideration may be an entire System [2]. In practice, the System is broken down to Subsystems and elements whose individual Reliability factors can be estimated or determined. Depending on the manner in which these Subsystems and elements are connected to constitute the given System, the combinatorial rule of probability is applied to obtain the System Reliability [8].

The term Dynamic Programming was originally used in the 1940s by Richard Bellman to describe the process of solving problems where one needs to find the best decisions, one after another. By 1953, he refined this to the modern meaning, referring specifically to nesting smaller decision problems inside larger decisions, and the field was thereafter recognized by the IEEE as a Systems Analysis and Engineering topic. Bellman's contribution is remembered in the name of the Bellman equation, a central result of Dynamic Programming which restates an Optimization problem in recursive form [3]. He applied this to a wide variety of problems. The general approach of Dynamic Programming is that, decision making is usually made in a sequence of stages, each stage having its own parameter of constraints called state. [11]

In Mathematics, Dynamic Programming is a method for solving complex problems by breaking them down into simpler sub-problems. It is applicable to problems exhibiting the properties of overlapping sub-problems, which are only slightly smaller and optimal substructure. When applicable, the method takes far less time than naïve methods. The key idea behind Dynamic Programming is quite simple. In general, to solve a given problem, we need to solve different parts of the problem (sub-problems) and then combine the solutions of the sub problems to reach an overall solution. Often, many of these sub-problems are really the same. The Dynamic Programming approach seeks to solve each sub-problem only once, thus reducing the number of computations. This is especially useful when the number of repeating sub-problems is exponentially large. The heart of the Dynamic Programming approach is the principle of optimality set forth by Bellman. It states that an optimal policy has the property that whatever the initial stage and the initial decisions are the remaining decisions must constitute an optimal policy with regard to the state resulting from first decision.

The basic features which characterize dynamic programming problems are presented and discussed below[11]

1) The problem can be divided into stages, with a policy decision required at each stage.Each stage has a number of states associated with it.
2) The effect of the policy decision at each stage is to transform the current state into a state associated with the next stage according to the probability distribution.
3) Given the current state, an optimal policy for the remaining stages is independent of the policy adopted in previous stages.
4) The solution procedure begins by finding the optimal policy for each state of last stage.
5) A recursive relationship that identifies optimal policy for each state at stage n, given the optimal policy for each state at stage (n+1), is available.

In this paper we make a Study of a Reliability Model with Series-Parallel Subsystems, where the Redundancy strategy is chosen for individual Subsystems. For this, we have made use of Dynamic programming Technique of Operations Research to determine the maximum Reliability with minimum cost allocated to the System. Comparison Study has been made with the System Reliability determined using Fault Tree Analysis Approach with that of the System Reliability obtained using Dynamic Programming Technique with the consideration that the cost allocated to the system is to be minimum.

The paper is organised as follows. In section 2, we give the basic concepts and definitions. In section 3, we introduced Series-Parallel System and determine its Reliability Using Fault Tree Analysis Approach. In section 4, we give the Reliability model using Dynamic Programming Technique. We have discussed in section 5 this problem with a Medical example and a Comparative Study has been made. In section 6, we draw the conclusions.

2. Basic Definitions and Concepts

2.1 Reliability

Reliability is defined as the probability of a device (or an item) performing its purpose for the period intended under the given operating conditions. It can also be defined as the probability of non-failure. If F(t) is the failure probability, then $1-F(t)$ gives the non-failure probability. Thus, the Reliability of the device (or an item) for time T = t (ie., the device functions satisfactorily for time $T \geq t$ is

$$R(t) = 1 - F(t)$$

$$= 1 - \int_{-\infty}^{t} f(x)\,dx$$

or $R(t) = \int_{t}^{\infty} f(x)\,dx$

where, $R(t)$ is the Reliability at time t

2.2 Series System

In a Series System, all Components in the System should be operating to maintain the required operation of the System. Thus, the failure of any one Component of the System will cause failure of the whole System.

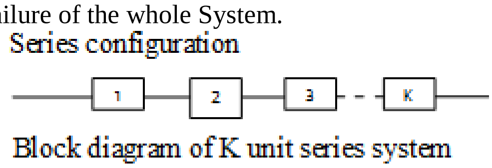

Fig. 1

The System Reliability Estimate of the Series System is given by

$$\hat{R}_s = \prod_{i=1}^{m} P(X_i)$$

where $P(X_i)$ is the Reliability of the i^th component

2.3 Parallel System

In a Parallel System, the System operates if one or more Components operate, and the System fails if all the Components fail. The Parallel n-components are represented by the following block diagram.

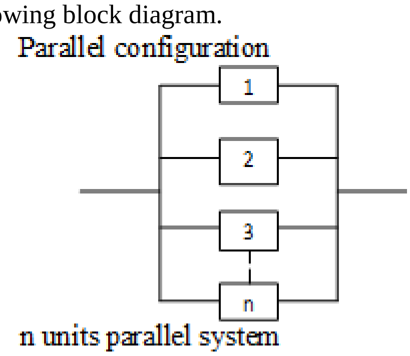

Fig. 2

$$R_p(t) = 1 - \prod_{i=1}^{n} \left(1 - P(X_i)\right)$$

where $P(X_i)$ is the probability of the i^th component.

2.4 Redundancy

We can improve the System Reliability by the technique of introducing Redundancies. This involves the deliberate creation of the new Parallel paths in the System. We have observed that if two elements with probabilities of success P (a) and P (b) are connected in Parallel, the probability P (a or b) is

$$P(a\,or\,b) = P(a) + P(b) - P(a\,and\,b)$$

$$= P(a) + P(b) - P(a) \times P(b)$$

Assuming that the elements are independent, since both P (a) and P (b) are individually less than one, their product is always less than P (a) or P (b). Hence, $P(a\,or\,b)$ is always greater than either P (a) or P(b).Although either one of the elements is sufficient for the successful operation of the system, we deliberately use both elements so as to increase the Probability of Success, thus causing the System to become Redundant.

2.5 Fault tree Analysis (FTA) approach

FTA is a top-down approach of a system analysis that is used to determine the possible occurrence of undesirable events or failures. Over the years, the method has gained favor over other reliability analysis approaches because of its versatility in degree of detail of complex systems. There are many symbols used to construct fault trees. The basic four symbols are

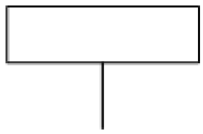

Figure 3

It denotes a fault event that occurs from the logical combination of fault events through the input of logic gates such as OR and AND

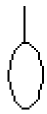

Figure 4

It denotes a basic fault event

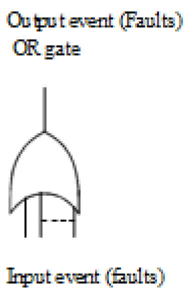

Figure 5

It denotes the output fault event if one or more of input fault events occur.

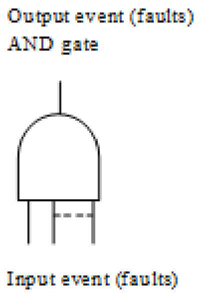

Figure 6

It denotes that an output fault tree event occurs if all the input fault events occur

3. General Series Parallel System

The System consists of stage 1, stage 2,…,stage k connected in Series. Each stage contains a
number of Redundant elements,

Stage i consisting of n_i Redundant elements connected in Parallel. The Reliability of the System is the product of the Reliabilities of each stage. Stage i with n_i will have the Reliability, $R_i = 1-[1-P(X_{i1})][1-P(X_{i2})],...,[1-P(X_{in_i})]$

$$=1-\prod_{j=1}^{n_i}\left[1-P\left(X_{ij}\right)\right]$$

Therefore, the System Reliability is

$$R(s)=R_1R_1R_1...R_1$$

$$=\prod_{i=1}^{k}\left\{1-\prod_{j=1}^{n_i}\left[1-P(X_{ij})\right]\right\} \quad .... (1)$$

Now, for Example, Consider a system which has 2 subsystems connected in series (say X_1 and X_2) each system has 2 components connected in parallel (say X_{11} and X_{12}). The Fault ttree analysis diagram of the system is given below,

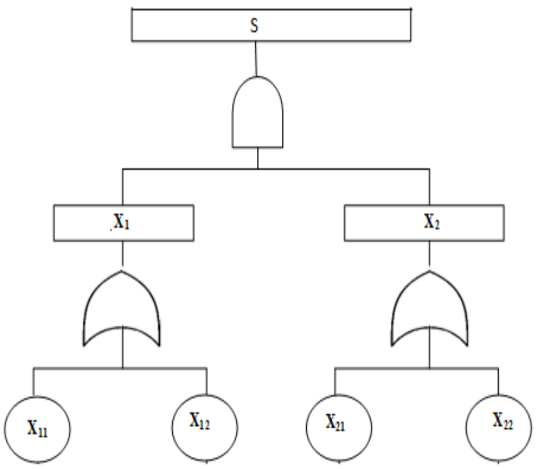

Figure 7

Dynamic Programming Formulation to a Reliability Model

Reliability is a most important requirement for many Medical Systems, such as those designed for multistage operation systems. Such Systems can be considered as a series of "black boxes" or subsystems. If by chance a subsystem fails, the entire System fails. To avoid chance of failure, that is to say, increase the Reliability of the whole System, Redundancy in some or all of the Systems is incorporated. This means that instead of using just one Subsystems in stage j, two or more are connected in parallel along with switching circuits in such a way that if one operating Subsystem fails, one of the Redundant Subsystem is automatically switched on.

Notations

c_j - Cost of one subsystem in stage j.

M -Total Money available for the entire system.

$1+x_j$ -The number of subsystems in stage j.

$p_j(x_j)$ - The probability that the j[th] stage operates successfully.

x_j - is the redundancy in the stage

where $x_j \geq 0, x_j \leq m_j$ some given number.

Assuming that $p_j(x_j)$ is independent of what is done for other stages, the reliability of the whole system is $p_1(x_1)\times p_2(x_2)\times ...\times p_n(x_n)$.The total cost involved is $\sum_{j=1}^{n} c_j\left(1+x_j\right)$.

Application To a Reliability Model

x_j	$p_1(x_1)$	$p_2(x_2)$	$p_3(x_3)$
0	0.6	0.8	0.7
1	0.7	0.8	0.8
2	0.9	0.9	0.9

Hence for Maximum Reliability, we have the problem Maximise:

$$z=p_1(x_1)\times p_2(x_2)\times...\times p_n(x_n) \quad ………(2)$$

$$\text{s.t} \sum_{j=1}^{n} c_j x_j \leq M-\sum_{j=1}^{n} c_j = b \text{ (say)}$$

$$\text{Integer } x_j \geq 0 \quad and \quad \leq m_j$$

The optimal values in the 'n' stages are given by the recursive relations

$$\xi_{k-1}=\xi_k - c_k x_k \text{ ,(k=n,n-1,…,2) with } \xi_n = b \quad ....(3)$$

Now,

$$F_k\left(\xi_k\right)=\max_{0\leq x_k\leq \min\{m_k,[\xi_k/c_k]\}}\left\{f_k\left(x_k\right)\times F_{k-1}\left(\xi_k - c_k x_k\right)\right\}$$

Forn k= n, n-1 ,…,2 and

$$F_1\left(\xi_1\right)=\max_{0\leq x_1\leq \min\{m_1,[\xi_1/c_1]\}}\left\{f_1\left(x_1\right)\right\},\ \xi_1=0,1,2,...,b.$$

The Maximum Reliability is $F_n\left(\xi_n\right)$. Recursion is started and carried through the stages 2,3,…,n.

4. A Medical Example

A Health Organization conducts a free treatment scheme for a System consisting of three category of patients (regarded as subsystems) infected by a particular disease. The three patients are under three different stages of the disease (less affected, moderately affected, more affected). Three Doctors (Redundancies of the subsystems) are available for treatment of these patients with per unit cost of the three categories of patients as 1, 3 and 2 respectively. The probability of the subsystems are given below

If the total allocation cannot exceed Rs.1, 30,000, We determine the redundancies for the most reliable system. The mathematical model is

$$\text{Maximize: } z = p_1(x_1)\times p_2(x_2)\times p_3(x_3)$$

$$s.t\ \ 1x_1 + 3x_2 + 2x_3 = 13-(1+3+2) = 7$$

$$Integer\ x_j \geq 0\ \ and \leq 2$$

From equation (1) we know that $\xi_n = b$

Choosing n=3 we have $\xi_3 = 7$

The recurrence relations in Equation (1) are

$$\xi_2 = 7 - 2x_3$$

$$\xi_1 = 7 - 3x_2 - 2x_3$$

$$= \xi_2 - 3x_2$$

The Admissible values of ξ_2 *and* ξ_1 are

x_3	0	1	2
ξ_2	7	5	3

and

x_2	0	0	0	1	1	1	2
x_3	0	1	2	0	1	2	0
ξ_1	7	5	3	4	2	0	1

At the first stage the optimal value is given by

$$F_1(\xi_1) = \max_{0\leq x_1 \leq \min\{2,[\xi_1/1]\}} \{f_1(x_1)\},\ \xi_1 = 0,1,2,3,4,5,7. \qquad .... (4)$$

Now take ξ_1 =0 and x_1 =0, 1, 2

$$F_1(0) = \max_{0\leq x_1 \leq \min\{2,0\}} \{f_1(x_1)\} = \max\{0.6\} = 0.6$$

At ξ_1 =1 and x_1 =0, 1, 2 we have

$$F_1(1) = \max_{0\leq x_1 \leq \min\{2,1\}} \{f_1(x_1)\} = \max\{0.6, 0.7\} = 0.7$$

At ξ_1 =2 and x_1 =0, 1, 2 we have

$$F_1(2) = \max_{0\leq x_1 \leq \min\{2,2\}} \{f_1(x_1)\} = \max\{0.6, 0.7, 0.9\} = 0.9$$

The table below shows the first stage values.

	$f_1(x_1)$			
x_1 / ξ_1	0	1	2	$F_1(\xi_1)$
0	0.6			0.6
1	0.6	0.7		0.7
≥ 2	0.6	0.7	0.9	0.9

Similarly, at the second Stage

$$F_2(\xi_2) = \max_{0\leq x_2 \leq \min\{2,[\xi_2/2]\}} \{f_2(x_2)\times F_1(\xi_1)\},\ \xi_2 = 3,5,7.$$

The table for this is given below

	$f_2(x_2)\times F_1(\xi_2 - 3x_2)$			
x_2 / ξ_2	0	1	2	$F_2(\xi_2)$
3	0.8 x 0.9	0.8 x0.6		0.72
5	0.8 x 0.9	0.8 x 0.7		0.72
7	0.8 x 0.9	0.8 x 0.9	0.9 x 0.7	0.72

At the final Stage

$$F_3(\xi_3) = \max_{0\leq x_3 \leq \min\{2,[\xi_3/2]\}} \{f_2(x_2)\times F_1(\xi_1)\},\ \xi_3 = 7.$$

For which we have the table

	$f_3(x_3)\times F_2(7-2x_3)$			
x_3 x_3 / ξ_3	0	1	2	$F_3(\xi_3)$
7	0.7x0.72	0.8x0.72	0.9 x 0.72	0.648

From the table, we get the Maximum is for $x_3 = 2$ and by trace back $x_2 = 0$, $x_1 = 2$.The maximum Reliability is 0.648 at a cost of $3\times1 + 1\times3 + 3\times2 = 12$ (in tens of lakhs of rupees).

But from the equation (1) Reliability of the System for the given numerical example is determined as 0.9633. This clearly shows that when the Cost of the System is taken into consideration, we find that the Reliability of the System is reduced by 31 percent.

5. Conclusion

Dynamic programming is a newly developed Mathematical technique which is often useful for making a sequence of inter-related decisions. We have made use of this technique to allocate the minimum resource for a System and find its Reliability. Comparative study tells us that the System Reliability decreases when Cost of the System is taken care off, where we have considered the Cost of the System not to exceed the Total money allocated for the entire System.

6. Acknowledgement

This paper is funded by the UGC under Minor Research Project.

References

[1] Balagurusamy.E., Reliability Engineering, Tata Mc Graw-Hill Publishing Company Limited, 2010.
[2] Chang,T., System Reliability analysis with Dynamic Programming.
[3] Eddy, S.R ., What is dynamic programming ?, Nature Biotechnology, 22,909-910(2004).
[4] Fredrick S.Hillier., Gerald J., Lieberman., Introduction To Operation Research, Second Edition, Mc Graw Hill publications
[5] HamdyA.Taha., Operations Research: An Introduction 8th Edition, Pearson prentice hall, Pearson Education,Inc
[6] Kasana H. S., Kumar K. D., Advance Operations Research, Asian Book Publisher ,2005.
[7] Prem Kumar Gupta., D.S. Hira., Operations Research 6th edition.
[8] Srinath L.S., Reliability Engineering fourth edition, Affiliated East- West Press Private Limited
[9] Sujit K Bose ., Operations Research Methods Narosa Publishing House , New Delhi.
[10] Sharma.J.K.,Operations Research: theory and application, Macmillan Publishers.
[11] F.A. Tillman, C.V. Hwang & W. Kuo, Optimization of Systems Reliability, Cambridge University Press, 2001.