On Pre- $\rho$ -Continuity Where $\rho \in \{L, M, R, S\}$

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Abstract: The authors Selvi.R, Thangavelu.P and Anitha.M introduced the concept of $\rho$ -continuity between a topological space and a non empty set where $\rho \in \{L, M, R, S\}$ [4]. Navpreet singh Noorie and Rajni Bala[3] introduced the concept of $f^{\rho}$ function to characterize the closed, open and continuous functions. In this paper, the concept of pre-$\rho$ -continuity is introduced and its properties are investigated and pre-$\rho$ -continuity is further characterized by using $f^{\rho}$ functions.

Keywords: Multifunction, Saturated set, $\rho$ -continuity, pre-open, pre-closed and continuity

1. Introduction

By a multifunction $F: X \rightarrow Y$, We mean a point to set correspondence from $X$ into $Y$ with $F(x) \neq \emptyset$ for all $x \in X$. Any function $f: X \rightarrow Y$ induces a multifunction $f^{-1}$ of $X \rightarrow \mathcal{P}(X)$. It also induces another multifunction $f \circ f^{-1}: Y \rightarrow (Y)$ provided $f$ is surjective. The purpose of this paper is to introduce notions of pre-L-Continuity, pre- M-Continuity, pre-R-Continuity and pre-S-Continuity of a function $f: X \rightarrow Y$ between a topological space and a non empty set. Here we discuss their links with pre-open and pre-closed sets. Also we establish pasting lemmas for pre-R-continuous and pre-S-continuous functions. In an analogous way pre-$\rho$-continuity is characterized in this paper.

2. Preliminaries

The following definitions and results that are due to the authors [4] and Navpreet singh Noorie and Rajni Bala [3] will be useful in sequel.

Definition 2.1
Let $f: (x, t) \rightarrow Y$ be a function. Then $f$ is
(i) L-Continuous if $f^{-1}(f(A))$ is open in $X$ for every open set $A$ in $X$. [4]
(ii) M-Continuous if $f^{-1}(f(A))$ is closed in $X$ for every closed set $A$ in $X$. [4]

Definition 2.2
Let $f: (X, t) \rightarrow (Y, \sigma)$ be a function. Then $f$ is
(i) R-Continuous if $f^{-1}(f^{-1}(B))$ is open in $Y$ for every open set $B$ in $Y$. [4]
(ii) S-Continuous if $f^{-1}(f^{-1}(B))$ is closed in $Y$ for every closed set $B$ in $Y$. [4]

Definition 2.3
Let $f: X \rightarrow Y$ be any map and $E$ be any subset of $X$. then the following hold. (i) $f^{\rho}(E) = \{y \in Y: f^{-1}(y) \subseteq E\}$ ; (ii) $E^\rho = f^{-1}(f^{\rho}(E))$. [3]

Lemma 2.4:
Let $E$ be a subset of $X$ and let $f: X \rightarrow Y$ be a function. Then the following hold. (i) $f^{\rho}(E) = Y \cap f^{-1}(f(X \cap E))$ ; (ii) $f(E) = Y \cup f^{\rho}(X \cap E)$. [3]

Lemma 2.5:
Let $E$ be a subset of $X$ and let $f: X \rightarrow Y$ be a function. Then the following hold. (i) $f^{-1}(f^{\rho}(E)) = X \cap f^{-1}(f(X \cap E))$ ; (ii) $f^{-1}(f(E)) = X \setminus f^{-1}(f^{\rho}(X \cap E))$. [6]

Lemma 2.6:
Let $E$ be a subset of $X$ and let $f: X \rightarrow Y$ be a function. Then the following hold. (i) $f^{\rho}(f^{-1}(Y)) = Y \cap f^{-1}(f(X \cap E))$ ; (ii) $f^{-1}(f(E)) = Y \cup f^{\rho}(f^{-1}(Y \cap E))$. [6]

Definition 2.7
Let $f: X \rightarrow Y$, $A \subseteq X$ and $B \subseteq Y$. we say that $A$ is $f$ - saturated if $f^{-1}(f(A)) \subseteq A$ and $B$ is $f^{\rho}$-saturated if $f^{-1}(f^{\rho}(A)) \subseteq B$. Equivalently $A$ is $f$-saturated if and only if $f^{-1}(f(A)) = A$, and $B$ is $f^{\rho}$-saturated if and only if $f^{-1}(f^{\rho}(B)) = B$.

Definition 2.8
Let $A$ be a subset of a topological space $(X, t)$. Then $A$ is called (i) semi-open if $A \subseteq \text{cl}(\text{int}(A))$ and semi-closed if $\text{int}(\text{cl}(A)) \subseteq A$; [1]. (ii) pre-open if $A \subseteq \text{int}(\text{cl}(A))$ and pre-closed if $\text{cl}(\text{int}(A)) \subseteq A$; [2].

Definition 2.9:
Let $f: (X, t) \rightarrow (Y, \sigma)$ be a function. Then $f$ is pre-continuous if $f^{-1}(B)$ is open in $X$ for every pre-open set $B$ in $Y$. [2]
Definition: 2.10:
Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then $f$ is pre-open (resp. pre-closed) if $f(A)$ is pre-open (resp. pre-closed) in $Y$ for every pre-open (resp. pre-closed) set $A$ in $X$.

3. Pre-\(\rho\) -Continuity Where \(\rho \in \{L, M, R, S\}\)

Definition: 3.1
Let $f: (X, \tau) \to Y$ be a function. Then $f$ is
(i) pre-$L$-Continuous if $f^{-1}(f(A))$ is open in $X$ for every pre-open set $A$ in $X$.
(ii) pre-$M$-Continuous if $f^{-1}(f(A))$ is closed in $X$ for every pre-closed set $A$ in $X$.

Definition: 3.2
Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then $f$ is
(i) pre-$R$-Continuous if $f^{-1}(f(A))$ is open in $Y$ for every pre-open set $B$ in $Y$.
(ii) pre-$S$-Continuous if $f^{-1}(f(A))$ is closed in $Y$ for every pre-closed set $B$ in $Y$.

Example: 3.3
Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$. Let $f = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $g: (Y, \sigma) \to (\{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$. Let $g: (X, \tau) \to (Y, \sigma)$ be defined by $g(a)=1, g(b)=2, g(c)=3, g(d)=4$. Then $f$ is pre-$L$-Continuous and pre-$M$-Continuous.

Definition: 3.5
Let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then $f$ is
(i) pre-$LR$-Continuous, if it is both pre-$L$-Continuous and pre-$R$-Continuous.
(ii) pre-$LS$-Continuous, if it is both pre-$L$-Continuous and pre-$S$-Continuous.
(iii) pre-$MR$-Continuous, if it is both pre-$M$-Continuous and pre-$R$-Continuous.
(iv) pre-$MS$-Continuous, if it is both pre-$M$-Continuous and pre-$S$-Continuous.

Theorem: 3.6
(i) Every injective function $f: (X, \tau) \to (Y, \sigma)$ is pre-$L$-Continuous and pre-$M$-Continuous.
(ii) Every surjective function $f: (X, \tau) \to (Y, \sigma)$ is pre-$R$-Continuous and pre-$S$-Continuous.
(iii) Any constant function $f: (X, \tau) \to (Y, \sigma)$ is pre-$R$-Continuous and pre-$S$-Continuous.

Proof:
(i) Let $f: (X, \tau) \to (Y, \sigma)$ be injective function. Then pre-$L$-Continuity and pre-$M$-Continuity follow from the fact that $f^{-1}(f(A)) = A$. This proves (i).
(ii) Let $f: (X, \tau) \to (Y, \sigma)$ be surjective function. Since $f$ is surjective, $f^{-1}(B) = Y$ for every subset $B$ of $Y$. Then $f$ is both pre-$R$-Continuous and pre-$S$-Continuous. This proves (ii).
(iii) Suppose $f(x) = y_0$ for every $x$ in $X$. Then $f^{-1}(B) = Y$ if $y_0 \in B$ and $f^{-1}(B) = \emptyset$, if $y_0 \notin Y \cap B$. This proves (iii).

Corollary: 3.7
If $f: (X, \tau) \to (Y, \sigma)$ be bijective function then $f$ is pre-$L$-Continuous, pre-$M$-Continuous, pre-$R$-Continuous and pre-$S$-Continuous.

Theorem: 3.8
Let $f: (X, \tau) \to (Y, \sigma)$. (i) If $f$ is $L$-Continuous (resp. $M$-Continuous) then it is pre-$L$-Continuous (resp. pre-$M$-Continuous).
(ii) If $f$ is $R$-Continuous (resp. $S$-Continuous) then it is pre-$R$-Continuous (resp. pre-$S$-Continuous).

Proof:
(i) Let $A \subseteq X$ be pre-open (resp. pre-closed) in $X$. Since every pre-open (resp. pre-closed) set is open (resp. closed) and since $f$ is $L$-continuous (resp. $M$-continuous), $f^{-1}(f(A))$ is open (resp. closed) in $X$. Therefore $f$ is pre-$L$-Continuous (resp. pre-$M$-Continuous).
(ii) Let $B \subseteq Y$ be pre-open (resp. pre-closed) in $Y$. Since every pre-open (resp. pre-closed) set is open (resp. closed) and since $f$ is $R$-continuous (resp. $S$-continuous), $f^{-1}(f(B))$ is open (resp. closed) in $Y$. Therefore $f$ is pre-$R$-Continuous (resp. pre-$S$-Continuous).

Theorem: 3.9
Let $f: (X, \tau) \to (Y, \sigma)$. Then $int(cl(A))$ is $f$-saturated whenever $A$ is $f$-saturated and semi-closed.

Proof:
Let $A \subseteq X$ be $f$-saturated. Since $f$ is pre-$L$-Continuous $\Rightarrow A$ is pre-open set in $X \Rightarrow (A \subseteq cl(A))$ and since $A$ is semi-closed $\Rightarrow int(cl(A)) \subseteq A$. Therefore $int(cl(A)) = A$. Since $A$ is $f$-saturated $\Rightarrow f^{-1}(f(A)) = A$. That implies $int(cl(A)) = f^{-1}(int(cl(A)))$. Therefore $int(cl(A))$ is $f$-saturated whenever $A$ is $f$-saturated and semi-closed.

Theorem: 3.10
Let $f: (X, \tau) \to (Y, \sigma)$ be pre-$M$-Continuous. Then $cl(int(A))$ is $f$-saturated whenever $A$ is $f$-saturated and semi-open.

Proof:
Let $A \subseteq X$ be $f$-saturated. Since $f$ is pre-$M$-Continuous $\Rightarrow A$ is pre-closed set in $X \Rightarrow cl(int(A)) \subseteq A$ and since $A$ is semi-open $\Rightarrow A \subseteq cl(int(A))$. Therefore $cl(int(A)) = A$. Since $A$ is $f$-saturated $\Rightarrow f^{-1}(f(A)) = A$. That implies $cl(int(A)) = f^{-1}(f(cl(int(A))))$. Hence $cl(int(A))$ is $f$-saturated whenever $A$ is $f$-saturated and semi-open.

Theorem: 3.11
Let $f: X \to (Y, \sigma)$ be pre-$R$-Continuous. Then $int(cl(B))$ is $f^{-1}$-saturated whenever $B$ is $f^{-1}$-saturated and semi-closed.

Proof:
Let $B \subseteq Y$ be $f^{-1}$-saturated. Since $f$ is pre-$R$-Continuous $\Rightarrow B$ is pre-open set in $Y \Rightarrow int(cl(B)) \subseteq B$, and since $B$ is semi-closed $\Rightarrow int(cl(B)) \subseteq B$. Therefore $int(cl(B)) = B$. Since $B$ is $f^{-1}$-saturated $\Rightarrow f^{-1}(f^{-1}(B)) = B$ which implies that $f^{-1}(f^{-1}(cl(B))) = int(cl(B))$. Therefore $int(cl(B))$ is $f^{-1}$-saturated.

Theorem: 3.12
Let $f: X \to (Y, \sigma)$ be pre-$S$-Continuous Then $cl(int(B))$ is $f^{-1}$-saturated whenever $B$ is $f^{-1}$-saturated and semi-open.
4. Properties

In this section we prove certain theorems related with pre-open and pre-closed functions.

**Theorem 4.1**

(i) Let \( f: (X, \tau) \to (Y, \sigma) \) be open and pre-Continuous. Then \( f \) is pre-L-Continuous, pre-R-Continuous. (ii) Let \( f: (X, \tau) \to (Y, \sigma) \) be closed and pre-Continuous, Then \( f \) is pre-S-Continuous.

Proof:

(i) Let \( f: (X, \tau) \to (Y, \sigma) \) be open and pre-Continuous. Since \( f \) is open \( \Rightarrow \) \( f^{-1}(B) \) is open in \( Y \) and since \( f \) is pre-continuous \( \Rightarrow f^{-1}(f(A)) \) is open in \( X \). Therefore \( f \) is pre-L-Continuous. This proves (i).

(ii) Let \( f: (X, \tau) \to (Y, \sigma) \) be closed and pre-Continuous. Since \( f \) is closed \( \Rightarrow f^{-1}(B) \) is open in \( X \) and since \( f \) is open \( \Rightarrow f^{-1}(f^{-1}(B)) \) is open in \( Y \). Therefore \( f \) is pre-S-Continuous. This proves (ii).

**Theorem 4.2**

(i) Let \( f: (X, \tau) \to (Y, \sigma) \) be pre-open and pre-Continuous. Then \( f \) is pre-M-Continuous. (ii) Let \( f: (X, \tau) \to (Y, \sigma) \) be closed and pre-Continuous. Then \( f \) is pre-S-Continuous.

Proof:

(i) Let \( f: (X, \tau) \to (Y, \sigma) \) be pre-open and pre-Continuous. Since \( f \) is pre-open \( \Rightarrow f^{-1}(f(A)) \) is open in \( Y \) and since \( f \) is pre-continuous \( \Rightarrow f^{-1}(f(A)) \) is open in \( X \). Therefore \( f \) is pre-M-Continuous. This proves (i).

(ii) Let \( f: (X, \tau) \to (Y, \sigma) \) be closed and pre-Continuous. Since \( f \) is closed \( \Rightarrow f^{-1}(B) \) is closed in \( X \) and since \( f \) is open \( \Rightarrow f^{-1}(f^{-1}(B)) \) is closed in \( Y \). Therefore \( f \) is pre-S-Continuous. This proves (ii).

**Theorem 4.3**

Let \( X \) be a topological space.

(i) If \( A \) is a pre-open subspace of \( X \), the inclusion function \( j: A \to X \) is pre-M-continuous and pre-R-continuous. (ii) If \( A \) is a pre-closed subspace of \( X \), the inclusion function \( j: A \to X \) is pre-M-continuous and pre-S-continuous.

Proof:

(i) Suppose \( A \) is a pre-open subspace of \( X \). Let \( j: A \to X \) be an inclusion function. Let \( U \subseteq X \) be pre-open in \( X \) then \( j^{-1}(U) = U \cap A \subseteq A \) which is open in \( X \). Hence \( j \) is pre-R-continuous. Now, let \( U \subseteq A \) be pre-open in \( A \). Then \( j^{-1}(j(U)) = j^{-1}(U) = U \) which is open in \( A \). Hence \( j \) is pre-M-continuous. This proves (i).

(ii) Suppose \( A \) is a pre-closed subspace of \( X \). Let \( j: A \to X \) be an inclusion function. Let \( U \subseteq X \) be pre-closed in \( X \) then \( j^{-1}(U) = U \cap A \subseteq A \) which is closed in \( X \). Hence \( j \) is pre-S-continuous. Now, let \( U \subseteq A \) be pre-closed in \( A \). Then \( j^{-1}(j(U)) = j^{-1}(U) = U \) which is closed in \( A \). Hence \( j \) is pre-M-continuous. This proves (ii).

**Theorem 4.4**

Let \( g: Y \to Z \) and \( f: X \to Y \) be any two functions. Then the following hold. (i) If \( g: Y \to Z \) is pre-L-continuous (resp. pre-M-continuous) and \( f: X \to Y \) is pre-open (resp. pre-closed) and continuous, then \( g f: X \to Z \) is pre-L-continuous (resp. pre-M-continuous). (ii) If \( g: Y \to Z \) is open (resp. closed) and pre-continuous and \( f: X \to Y \) is R-continuous (resp. S-continuous), then \( g f: X \to Z \) is pre-R-continuous (resp. pre-S-continuous).

Proof:

(i) Suppose \( g \) is pre-L-continuous (resp. pre-M-continuous) and \( f \) is pre-open (resp. pre-closed) and continuous. Let \( A \subseteq X \) be pre-open (resp. pre-closed) in \( X \). Then \( (g f)^{-1}(g f)(A) = f^{-1}(g f)(A) \subseteq f^{-1}(g f)(A) \subseteq f^{-1}(B) \). Since \( f \) is pre-open (resp. pre-closed) \( \Rightarrow f^{-1}(f(A)) \) is open (resp. pre-closed) in \( Y \). Since \( g \) is pre-L-continuous (resp. pre-M-continuous), \( \Rightarrow g^{-1}(g f)(A) \subseteq g^{-1}(g f)(A) \subseteq g^{-1}(B) \) is open (resp. pre-closed) in \( Z \). Since \( f \) is continuous \( \Rightarrow f^{-1}(f(A)) \) is open (resp. pre-closed) in \( Y \). Since \( g \) is continuous \( \Rightarrow g f: X \to Z \) is pre-L-continuous (resp. pre-M-continuous). This proves (i).

(ii) Suppose \( g \) is pre-L-continuous (resp. pre-M-continuous) and \( f \) is pre-closed (resp. pre-open) and continuous. Let \( B \subseteq Y \) be pre-closed (resp. pre-open) in \( Y \). Then \( (g f)^{-1}(g f)(B) = (g f)(f^{-1}(g f)(B)) \subseteq (g f)(f^{-1}(B)) \) is open (resp. pre-closed) in \( Y \). Since \( f \) is continuous \( \Rightarrow f^{-1}(f^{-1}(B)) \) is open (resp. pre-closed) in \( Z \). Since \( g \) is continuous \( \Rightarrow g f: X \to Z \) is pre-R-continuous (resp. pre-S-continuous). This proves (ii).

**Theorem 4.5**

If \( f: X \to Y \) is pre-L-continuous and if \( A \) is an open subspace of \( X \), then the restriction of \( f \) to \( A \) is pre-L-continuous.

Proof:

Let \( h = f|_A \). Then \( h = f o j \), where \( j \) is the inclusion map \( j: A \to X \). Since \( j \) is open and continuous and since \( f: X \to Y \) is pre-L-continuous, using theorem (4.4(i)), \( h: A \to Y \) is pre-L-continuous.

**Theorem 4.6**

If \( f: X \to Y \) is pre-M-continuous and if \( A \) is a closed subspace of \( X \), then the restriction of \( f \) to \( A \) is pre-M-continuous.

Proof:

Let \( h = f|_A \). Then \( h = f o j \), where \( j \) is the inclusion map \( j: A \to X \). Since \( j \) is closed and continuous and since \( f: X \to Y \) is pre-M-continuous, using theorem (4.4(ii)), \( h: A \to Y \) is pre-M-continuous.
Theorem: 4.7
Let \( f: X \rightarrow Y \) be pre-R-continuous. Let \( f(x) \subseteq Z \subseteq Y \) \ and \( f(X) \) be open in \( Z \). Let \( h: X \rightarrow Z \) be obtained by \( f \) by restricting the codomain of \( f \) to \( Z \). Then \( h \) is pre-R-continuous.

Proof:
Clearly \( h = j \circ f \) where \( j: f(x) \rightarrow Z \) is an inclusion map. Since \( f(X) \) is open in \( Z \), the inclusion map \( j \) is both open and pre-continuous. Then by applying theorem 4.4(ii), \( h \) is pre-R-continuous.

Theorem: 4.8
Let \( f: X \rightarrow Y \) be pre-S-continuous. Let \( f(x) \subseteq Z \subseteq Y \) \ and \( f(X) \) be closed in \( Z \). Let \( h: X \rightarrow Z \) be obtained by \( f \) by restricting the codomain of \( f \) to \( Z \). Then \( h \) is pre-S-continuous.

Proof:
Clearly \( h = j \circ f \) where \( j: f(x) \rightarrow Z \) is an inclusion map. Since \( f(X) \) is closed in \( Z \), the inclusion map \( j \) is both closed and pre-continuous. Then by applying theorem 4.4(ii), \( h \) is pre-S-continuous.

Theorem: 5.2
A function \( f: X \rightarrow Y \) is pre-M-continuous if and only if \( f^{-1}(f(A)) \) is open in \( X \) for every pre-open subset \( G \) of \( X \).

Proof:
Suppose \( f \) is pre-M-continuous. Let \( G \) be pre-open in \( X \). Then \( A = X \setminus G \) is pre-closed in \( X \). Since \( f \) is pre-M-continuous and since \( A \) is pre-closed in \( X \), \( f^{-1}(f(A)) \) is closed in \( X \). By lemma \((2.5)-(i)) \Rightarrow f^{-1}(f(A)) = f^{-1}(f(G)) = f^{-1}(f(X \setminus G)) = f^{-1}(G) \) is open in \( X \). Conversely, we assume that \( f^{-1}(f(A)) \) is open in \( X \). By our assumption, \( f^{-1}(f(A)) \) is closed in \( X \). By lemma \((2.6)(ii)) \Rightarrow f^{-1}(f(A)) = \emptyset \) if \( f^{-1}(f(A)) \) is closed in \( X \). Therefore, \( f \) is pre-M-continuous.

Theorem: 4.9
Let \( X = A \cup B \). Let \( f: A \rightarrow (Y, \sigma) \) and \( g: B \rightarrow (Y, \sigma) \) be pre-R-continuous (resp. pre-S-continuous) \( f(x) = g(x) \) for every \( x \in A \cup B \), then \( h \) and \( g \) combined to give a pre-R-continuous (resp. pre-S-continuous) function \( h: X \rightarrow Y \) defined by \( h(x) = f(x) \) if \( x \in A \), and \( h(x) = g(x) \) if \( x \in B \).

Proof:
Let \( C \) be a pre-open (resp. pre-closed) set in \( Y \). Now \( hh^{-1}(C) = h(f^{-1}(C) \cup g^{-1}(C)) = h(f^{-1}(C)) \cup h(g^{-1}(C)) = h(f^{-1}(C)) \cup g(g^{-1}(C)) \). Since \( f \) is pre-R-continuous (resp. pre-S-continuous), \( f^{-1}(C) \) is open (resp. closed) in \( Y \) and since \( g \) is pre-R-continuous (resp. pre-S-continuous), \( g^{-1}(C) \) is open (resp. closed) in \( Y \). Therefore, \( hh^{-1}(C) \) is open (resp. closed) in \( Y \). Hence \( h \) is pre-R-continuous (resp. pre-S-continuous).

5. Characterizations

Theorem: 5.1
A function \( f: X \rightarrow Y \) is pre-L-continuous if and only if \( f^{-1}(f(A)) \) is closed in \( X \) for every pre-open subset \( A \) of \( X \).

Proof:
Suppose \( f \) is pre-L-continuous. Let \( A \) be pre-closed in \( X \). Then \( G = X \setminus A \) is pre-closed in \( X \). Since \( f \) is pre-L-continuous and since \( G \) is pre-closed in \( X \), \( f^{-1}(f(G)) \) is open in \( X \). By applying lemma \((2.5)-(i)) \Rightarrow f^{-1}(f(A)) = X \setminus f^{-1}(f(X \setminus A)) = X \setminus f^{-1}(f(G)) \). That implies \( f^{-1}(f(A)) \) is closed in \( X \). Conversely, we assume that \( f^{-1}(f(A)) \) is closed in \( X \) for every pre-closed subset \( A \) of \( X \). Let \( G \) be a pre-closed subset of \( X \). By our assumption, \( f^{-1}(f(A)) \) is closed in \( X \), where \( A = X \setminus G \). By using lemma \((2.5)-(ii)) \Rightarrow f^{-1}(f(A)) = X \setminus f^{-1}(f(X \setminus A)) = X \setminus f^{-1}(f(G)) \). That implies \( f^{-1}(f(A)) \) is open in \( X \). Therefore, \( f \) is pre-L-continuous.

Theorem: 5.3
The function \( f: X \rightarrow Y \) is pre-R-continuous if and only if \( f^{-1}(f(B)) \) is closed in \( Y \) for every pre-closed subset \( B \) of \( Y \).

Proof:
Suppose \( f \) is pre-R-continuous. Let \( B \) be pre-closed in \( Y \). Then \( G = Y \setminus B \) is open in \( Y \). Since \( f \) is pre-R-continuous and since \( G \) is pre-closed in \( Y \), \( f^{-1}(f(G)) \) is open in \( Y \). Now by using lemma \((2.6)(ii)) \Rightarrow f^{-1}(f(G)) = Y \setminus f^{-1}(f(Y \setminus G)) = Y \setminus f^{-1}(f(G)) \). That implies \( f^{-1}(f(B)) \) is closed in \( Y \). Conversely, we assume that \( f^{-1}(f(B)) \) is closed in \( Y \) for every pre-closed subset \( B \) of \( Y \). Let \( G \) be a pre-open subset of \( Y \). By our assumption, \( f^{-1}(f(G)) \) is open in \( Y \). Now by using lemma \((2.6)(ii)) \Rightarrow f^{-1}(f(G)) = Y \setminus f^{-1}(f(Y \setminus G)) = Y \setminus f^{-1}(f(G)) \). That implies \( f^{-1}(f(B)) \) is closed in \( Y \). Therefore, \( f \) is pre-R-continuous.

Theorem: 5.4
The function \( f: X \rightarrow Y \) is pre-S-continuous if and only if \( f^{-1}(f(B)) \) is open in \( Y \) for every pre-closed subset \( B \) of \( Y \).

Proof:
Suppose \( f \) is pre-S-continuous. Let \( B \) be pre-closed in \( Y \). Then \( G = Y \setminus B \) is open in \( Y \). Since \( f \) is pre-S-continuous and since \( B \) is pre-closed in \( Y \), \( f^{-1}(f(B)) \) is open in \( Y \). Now by using lemma \((2.6)(ii)) \Rightarrow f^{-1}(f(G)) = Y \setminus f^{-1}(f(Y \setminus G)) = Y \setminus f^{-1}(f(B)) \). That implies \( f^{-1}(f(B)) \) is open in \( Y \). Therefore, \( f \) is pre-S-continuous.

Theorem: 5.5
Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent. (i) \( f \) is pre-L-continuous, (ii) for every pre-closed subset \( A \) of \( X \), \( f^{-1}(f(A)) \) is closed in \( X \), (iii) for every \( x \in X \) and for every pre-open set \( U \) in \( X \) with \( f(x) \in f(U) \) there is an open set \( G \) in \( X \) with \( x \in G \) and \( f(G) \subseteq f(U) \), (iv) \( f^{-1}(f(int\{A\})) \subseteq f^{-1}(f(int\{A\})) \) for every semi-closed subset \( A \) of \( X \), (v) \( cl(f^{-1}(f(A))) \subseteq f^{-1}(f(cl(int\{A\})) \) for every semi-closed subset \( A \) of \( X \).
semi-open subset A of X. Proof: (i) \iff (ii) follows from theorem 5.1. (i) \iff (iii): Suppose f is pre-L-continuous. Let U be pre-open set in X such that f(x) \in f(U). Since f is pre-L-continuous, f^{-1}(f(U)) is open in X. Therefore f^{-1}(f(U)) \subseteq f(G) \subseteq f^{-1}(f(U)) \subseteq f(U). This proves (ii) conversely.

(iii): Suppose f is pre-L-continuous. Let U be pre-open set in Y and x \in f^{-1}(f(U)). Then f(x) \in f(U). Using (iii), there is an open set G in X containing x such that f(G) \subseteq f^{-1}(f(U)). Therefore x \in G \subseteq f^{-1}(f(G) \subseteq f^{-1}(f(U))). That implies f^{-1}(f(U)) is open set in X. This completes the proof for (i) \iff (iii).

(i) \iff (iv): Suppose f is pre-L-continuous. Let A be a semi-open subset of X. Then int(cl(A)) is pre-open set in X. Then int(cl(A)) is open in X. \Rightarrow f^{-1}(f(int(cl(A)))) \subseteq int(f^{-1}(f(int(cl(A)))). Since A is semi-open in X, we have f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(cl(int(A)))) \subseteq int(f^{-1}(f(cl(int(A))))). This proves (iv). Conversely, suppose (iv) holds. Let G be pre-open set in Y \Rightarrow f^{-1}(f(cl(int(A)))) \subseteq int(f^{-1}(f(cl(int(A))))). Since G is semi-closed in X, by using (iv) we get f^{-1}(f(cl(int(A)))) \subseteq int(f^{-1}(f(cl(int(A)))). Then it follows that f^{-1}(f(cl(int(A)))) is open in X. This proves (ii).

Theorem 5.7

Let f: X \to (Y, \sigma) be a function and \sigma be a space with a base consisting of f^{-1}saturated open sets. Then the following are equivalent. (i) f is pre-R-continuous, (ii) for every pre-closed subset B of Y, f^{-1}(f^{-1}(B)) is closed in Y. (iii) for every x \in X and for every pre-open set V in Y with f(x) \in f^{-1}(V) there is an open set G in Y with x \in G \subseteq f^{-1}(V).

Proof:

(i) \iff (ii): follows from theorem 5.3. (i) \iff (iii) Suppose f is pre-R-continuous. Let V be a pre-open set in Y such that x \in f^{-1}(V). Since f is pre-R-continuous, f^{-1}(f^{-1}(V)) is open in Y. f(x) \in f^{-1}(V) there is an open set G in Y such that f(x) \in G \subseteq f^{-1}(V). That implies x \in f^{-1}(G) \subseteq f^{-1}(f^{-1}(V)) \subseteq f^{-1}(V). This proves (iii). Conversely, suppose (iii) holds. Let V be pre-open in Y and y \in f^{-1}(V). Then y=f(x) for some x \in f^{-1}(V). By using (iii) there is an open set G in Y containing x such that f^{-1}(G) \subseteq f^{-1}(V). We choose G to be a f^{-1}saturated in Y. Then G=f^{-1}(f^{-1}(V)), which proves that f^{-1}(V) is open in Y. This proves that f is pre-R-continuous. (i) \iff (iv): Suppose f is pre-R-continuous. Let B be a semi-open subset of X. Then cl(f(int(B))) is pre-open set in Y. By the pre-R-continuity of f, we see that f^{-1}(f^{-1}(f(cl(int(B)))) is pre-open subset of Y. This proves (iv).

Theorem 5.6

Let f: (X, \tau) \to (Y, \sigma) be a function. Then the following are equivalent. (i) f is pre-M-continuous, (ii) for every pre-open subset G of X, f^{-1}(f^{-1}(G)) is open in Y. (iii) if f^{-1}(f^{-1}(G)) is closed in Y, then f^{-1}(f^{-1}(G)) \subseteq f^{-1}(f^{-1}(G)) for every semi-closed subset A of X. (iv) If f^{-1}(f^{-1}(G)) is closed in Y, then f^{-1}(f^{-1}(G)) \subseteq int(f^{-1}(f^{-1}(G))) for every semi-closed subset A of X.

Proof:

(i) \iff (ii): follows from theorem 5.2. (i) \iff (iii) Suppose f is pre-M-continuous. Let A be a semi-open subset in X. Then cl(int(A)) is pre-closed in X. Since f is pre-M-continuous \Rightarrow f^{-1}(f(cl(int(A)))) is closed in X. \Rightarrow f^{-1}(f(cl(int(A)))) = f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(cl(int(A)))) = f^{-1}(f(cl(int(A))))). This proves (iii). Conversely, suppose (iii) holds. Let A be a pre-open subset in X \Rightarrow f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(A)). Since A is semi-open by applying (iii), cl(f^{-1}(f(A))) \subseteq f^{-1}(f(cl(int(A)))) \subseteq f^{-1}(f(A)). That implies f^{-1}(f(A)) is closed in X. This completes the proof for (i) \iff (iii).
(v) holds. Let \( B \) be a pre-closed subset of \( Y \) \( \Rightarrow \) \( f^{-1}(\text{cl}(\text{int}(B)))) \subseteq f^{-1}(\text{cl}(B)) \), since \( f \) is semi-open in \( Y \), by (v), we see that \( \text{cl}(f^{-1}(\text{int}(B)))) \subseteq f^{-1}(\text{cl}(\text{int}(B)))) \Rightarrow \text{cl}(f^{-1}(\text{int}(B)))) \subseteq f^{-1}(\text{cl}(B)) \) \( \Rightarrow \) \( \text{cl}(\text{int}(B)))) \subseteq f^{-1}(\text{cl}(B)) \). Therefore \( f^{-1}(\text{cl}(B)) \) is closed in \( Y \). This proves (ii).

**Theorem 5.8**

Let \( f: X \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent.

1. \( f \) is pre-S-continuous,
2. for every pre-open subset \( V \) of \( Y \), \( f^{-1}(\text{int}(V)) \) is open in \( X \),
3. \( \text{cl}(f^{-1}(\text{cl}(B)))) \subseteq \text{cl}(f^{-1}(\text{cl}(B)))) \) for every semi-open subset \( B \) of \( Y \).
4. \( f^{-1}(\text{cl}(\text{int}(B)))) \subseteq \text{int}(f^{-1}(\text{cl}(B)))) \) for every semi-closed subset \( B \) of \( Y \).

Proof: (i) \( \Leftrightarrow \) (ii): follows from theorem 5.4. Let \( B \) be a semi-open set in \( Y \). Since \( f \) is pre-S-continuous, \( f^{-1}(\text{cl}(B)))) \) is closed in \( Y \). \( \Rightarrow \) \( \text{cl}(f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \). Since \( B \) is semi-open in \( Y \), we see that \( f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \Rightarrow \text{cl}(f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \). It follows that, \( \text{cl}(f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \) that implies \( f^{-1}(\text{cl}(B)))) \) is closed set in \( Y \). Conversely, suppose (iii) holds. Let \( B \) be pre-closed subset in \( Y \) \( \Rightarrow \) \( f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \). This completes the proof for (i) \( \Leftrightarrow \) (iii).

(ii) \( \Leftrightarrow \) (iv): Suppose (ii) holds. Let \( B \) be a semi-closed subset of \( Y \). Then \( \text{int}(\text{cl}(B)))) \) is pre-open in \( Y \). By (ii), \( f^{-1}(\text{cl}(B)))) \) is open in \( Y \) \( \Rightarrow \) \( f^{-1}(\text{cl}(B)))) \subseteq \text{int}(f^{-1}(\text{cl}(B)))) \). Since \( B \) is a semi-closed, it follows that \( f^{-1}(\text{cl}(B)))) \subseteq f^{-1}(\text{cl}(B)))) \Rightarrow \text{int}(f^{-1}(\text{cl}(B)))) \subseteq \text{int}(f^{-1}(\text{cl}(B)))) \Rightarrow f^{-1}(\text{cl}(B)))) \subseteq \text{int}(f^{-1}(\text{cl}(B)))) \Rightarrow \text{int}(f^{-1}(\text{cl}(B)))) \). Conversely, suppose (iv) holds. Let \( V \) be pre-open in \( Y \) \( \Rightarrow \) \( f^{-1}(\text{int}(V)))) \subseteq \text{int}(f^{-1}(\text{int}(V)))) \). Since \( V \) is semi-closed in \( Y \), by using (iv), we see that \( f^{-1}(\text{int}(V)))) \subseteq \text{int}(f^{-1}(\text{int}(V)))) \Rightarrow f^{-1}(\text{int}(V)))) \subseteq \text{int}(f^{-1}(\text{int}(V)))) \Rightarrow \text{int}(f^{-1}(\text{int}(V)))) \). Then it follows that \( \text{int}(f^{-1}(\text{int}(V)))) \) is open in \( Y \). This proves (ii).

6. Conclusion

In this paper the notions of Pre-L-Continuity, Pre-M-Continuity, Pre-R-Continuity and Pre-S-Continuity of a function \( f: X \rightarrow Y \) between a topological space and a non empty set are introduced. The purpose of this paper is to introduce, Pre-\( \rho \)-continuity. Here we discuss their links with Pre-open, Pre-closed sets. Also we establish pasting lemmas for Pre-R-continuous and Pre-s-continuous functions and obtain some characterizations for, Pre-\( \rho \)-continuity. We have put forward some examples to illustrate our notions.

**References**