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Vague Bi-ideals and Vague Quasi Ideals of a Γ-Semiring

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Abstract: ***The concepts of vague bi-ideal and vague quasi ideal of a Γ- semiring are introduced and we characterize regular Γ-semiring in terms of vague bi-ideals and vague quasi ideals.***

Keywords: Vague set, Left(Right) Vague ideal, Vague bi-ideal, Vague quasi ideal.

1. Introduction

In 1965, Zadeh.L.A[12] introduced the study of fuzzy sets. Mathematically a fuzzy set on a set X is a mapping μ into [0, 1] of real numbers; for x in X, μ(x) is called the membership of x belonging to X. The membership function gives only an approximation for belonging but it does not give any information of not belonging. To avoid this, Gau.W.L and Buehrer.D.J[4] introduced the concept of vague sets. A vague set A of a set X is a pair of functions (t_A, f_A), where t_A and f_A are fuzzy sets on X satisfying $t_A(x) + f_A(x) \leq 1$, for all x in X. A fuzzy set t_A of X can be identified with the pair $(t_A, 1 - t_A)$. Thus the theory of vague sets is a generalization of fuzzy sets. M.K.Rao[9] introduced the concept of Γ-semiring as a generalization of Γ-ring by Nobusawa.N[10] as well as semiring. After that Γ-semirings have been analyzed by many of the mathematicians. It is well known that ideals play an important role in any abstract algebraic structures. The properties of an ideal in semirings and Γ-semirngs were somewhat different from the properties of the usual ring ideals. Moreover the notion of Γ-semiring not only generalizes the notions of semiring and Γ-ring but also the notion of ternary semiring. However many authors like Jayanta Ghosh, Samanta.T.K, Dutta.T.K, Sujit Kumar Sardar were developed the theory of fuzzy Γ-semirings. Further on Γ-semirings, the study properties of fuzzy ideals, fuzzy prime ideals, fuzzy semiprime ideals and their generalizations play an important role in their structure theory. The notion of a bi-ideal was first introduced for semigroups by Good.R.A and Hughes.D.R[5]. After that Lajos.S and Szasz.F[7] initiated the idea of bi-ideals in a ring. In this paper, we introduce the concepts of vague bi-ideal and vague quasi ideal of a Γ-semiring and we proved intersection of a right vague ideal and a left vague ideal of a Γ-semiring is a vague quasi ideal of a Γ-semiring and we characterize regular Γ-semiring in terms of vague bi-ideals and vague quasi ideals.

2. Preliminaries

In this section we recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1: Let R and Γ be two additive commutative semigroups. Then R is called Γ-semiring if there exists a mapping $R \times \Gamma \times R \rightarrow R$ image to be denoted by aαb for a, b ∈ R and α ∈ Γ satisfying the following conditions.
1. $a\alpha(b + c) = a\alpha b + a\alpha c$
2. $(a + b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha + \beta)c = a\alpha c + a\beta c$
4. $a\alpha(b\beta c) = (a\alpha b)\beta c$, ∀ a, b, c ∈ R; α, β ∈ Γ.

Definition 2.2: A Γ-semiring R is said to be regular if for all x ∈ R, there exists a ∈ R and α, β ∈ Γ such that $x = x\alpha a\beta x$. A Γ-semiring R is said to be intra-regular if for all x ∈ R, there exists a, b ∈ R and α, β, γ ∈ Γ such that $x = a\alpha x\beta x\gamma b$.

Definition 2.3: A sub Γ-semiring S of a Γ-semiring R is said to be a bi-ideal of R if $S\Gamma R\Gamma S \subseteq S$. A nonempty subset S of a Γ-semiring R is said to be a quasi ideal of R if $(R\Gamma S)\cap(S\Gamma R) \subseteq S$.

Definition 2.4: Let X be any non-empty set. A mapping $\mu : X \rightarrow [0,1]$ is called a fuzzy subset of R.

Definition 2.5: A vague set A in the universe of discourse U is a pair (t_A, f_A), where $t_A : U \rightarrow [0, 1]$, $f_A : U \rightarrow [0, 1]$ are mappings such that $t_A(u) + f_A(u) \leq 1$, ∀ u ∈ U. The functions t_A and f_A are called true membership function and false membership function respectively.

Definition 2.6: Let A be a vague set of a universe U with true membership function t_A and false membership function f_A. For α, β ∈ [0,1] with α ≤ β, the (α, β)- cut or vague cut of a vague set A is the crisp subset of U is given by
$A_{(\alpha,\beta)} = \{x \in U / V_A(x) \geq [\alpha, \beta]\}$
i.e., $A_{(\alpha,\beta)} = \{x \in U / t_A(x) \geq \alpha \text{ and } 1 - f_A(x) \geq \beta\}$.

Definition 2.7: Let R_1 be a Γ_1-semiring and R_2 be a Γ_2-semiring. Then $(f, g) : (R_1, \Gamma_1) \rightarrow (R_2, \Gamma_2)$ is called a homomorphism if $f : R_1 \rightarrow R_2$ and $g : \Gamma_1 \rightarrow \Gamma_2$ are homomorphisms of semigroups such that
$f(x\gamma y) = f(x)g(\gamma)f(y)$, ∀ x, y ∈ R_1; γ ∈ Γ_1.

Definition 2.8: Let R be a Γ-semiring. A non empty set S of R is called an idempotent if SΓS = S. A vague set A of a Γ-semiring R is called idempotent if AΓA = A.

Definition 2.9[1]: Let R be a Γ-semiring. A vague set A = (t_A, f_A) on R is said to be vague Γ-semiring if the following conditions are true:
For all x, y ∈ R; γ ∈ Γ,
$V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \geq \min\{V_A(x), V_A(y)\}$
i.e., (i). $t_A(x + y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x + y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$ and
(ii). $t_A(x\gamma y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x\gamma y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$.

Definition 2.10[2]: A vague set A = (t_A, f_A) on R is said to be left (right) vague ideal of R if the following conditions are true:
For all x, y ∈ R; γ ∈ Γ,
$V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
$V_A(x\gamma y) \geq V_A(y)$ ($\geq V_A(x)$)
i.e., (i). $t_A(x + y) \geq \min\{t_A(x), t_A(y)\}$,
$1 - f_A(x + y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$ and
(ii). $t_A(x\gamma y) \geq t_A(y)$ ($\geq t_A(x)$),
$1 - f_A(x\gamma y) \geq 1 - f_A(y)$ ($1 - f_A(x\gamma y) \geq 1 - f_A(x)$).

3. Vague Bi-ideal of a Γ–Semiring

Throughout this paper unless otherwise mentioned R is a Γ-semiring and χ is the characteristic function of R. In this section we introduce the concept of vague bi-ideals of R and we characterize regular Γ-semiring in terms of vague bi-ideals.

Definition 3.1: A vague Γ-semiring A = (t_A, f_A) of R is called a vague bi-ideal of R if $V_A(x\alpha y\beta z) \geq \min\{V_A(x), V_A(z)\}$, ∀ x, y, z ∈ R; α, β ∈ Γ.
i.e., $t_A(x\alpha y\beta z) \geq \min\{t_A(x), t_A(z)\}$ and
$1 - f_A(x\alpha y\beta z) \geq \min\{1 - f_A(x), 1 - f_A(z)\}$

Definition 3.2: A vague Γ-semiring A = (t_A, f_A) of R is called a vague (1,2)-ideal of R if $V_A(x\alpha w\beta(y\gamma z)) \geq \min\{V_A(x), V_A(y), V_A(z)\}$, ∀ x, w, y, z ∈ R; α, β, γ ∈ Γ.
i.e., $t_A(x\alpha w\beta(y\gamma z)) \geq \min\{t_A(x), t_A(y), t_A(z)\}$ and
$1 - f_A(x\alpha w\beta(y\gamma z)) \geq \min\{1 - f_A(x), 1 - f_A(y), 1 - f_A(z)\}$,

Example 3.3: : Let R be the set of negative integers and Γ be the set of negative even integers. Then R, Γ are additive commutative semigroups.
Define the mapping R × Γ × R → R by aαb usual product of a, α, b, ∀ a, b ∈ R; α ∈ Γ.
Then R is a Γ-semiring.
Let A = (t_A, f_A), where $t_A : R \to [0, 1]$ and $f_A : R \to [0, 1]$ defined by

$$t_A(x) = \begin{cases} 0.5, \text{if } x = -1 \\ 0.7, \text{if } x = -2 \\ 0.9, \text{if } x < -2 \end{cases} \quad \text{and} \quad f_A(x) = \begin{cases} 0.5, \text{if } x = -1 \\ 0.2, \text{if } x = -2 \\ 0.1, \text{if } x < -2 \end{cases}$$

Then A is a vague bi-ideal of R.

Theorem 3.4: A necessary and sufficient condition for a vague Γ-semiring A = (t_A, f_A) of R to be a vague bi-ideal of R is that t_A and $1 - f_A$ are fuzzy bi-ideals of R.

Proof: Suppose A = (t_A, f_A) is a vague bi-ideal of R.
Let x, y, z ∈ R; α, β ∈ Γ.
By theorem:3.5[1], t_A and $1 - f_A$ are fuzzy Γ-semirings of R.
We have $V_A(x\alpha y\beta z) \geq \min\{V_A(x), V_A(z)\}$
i.e., $t_A(x\alpha y\beta z) \geq \min\{t_A(x), t_A(z)\}$
$1 - f_A(x\alpha y\beta z) \geq \min\{1 - f_A(x), 1 - f_A(z)\}$
Hence t_A and $1 - f_A$ are fuzzy bi-ideals of R.
The converse part is obvious from the definition.

Theorem 3.5: A vague set A = (t_A, f_A) of R is a vague bi-ideal of R if and only if for all α, β ∈ [0, 1], the vague cut or (α, β) - cut of A, $A_{(\alpha, \beta)}$ is a bi-ideal of R.

Proof: Suppose A is a vague bi-ideal of R.
From theorem:3.6[1], $A_{(\alpha, \beta)}$ is a sub Γ-semiring of R.
Let x, y ∈ $A_{(\alpha, \beta)}$, z ∈ R ; α, β ∈ Γ.
$\Rightarrow V_A(x) \geq [\alpha, \beta]$ and $V_A(y) \geq [\alpha, \beta]$
Now, $V_A(x\alpha z\beta y) \geq \min\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$
$\Rightarrow x\alpha z\beta y \in A_{(\alpha, \beta)}$.
Hence $A_{(\alpha, \beta)}$ is a bi-ideal of R.
Conversely suppose that $A_{(\alpha, \beta)}$ is a bi-ideal of R.
from theorem:3.6[1], A is a vague Γ-semiring of R.
Let x, y, z ∈ R and α, β ∈ Γ.
Let $V_A(x) = [\alpha_1, \beta_1]$ and $V_A(y) = [\alpha_2, \beta_2]$.
put $[\alpha, \beta] = \min\{[\alpha_1, \beta_1], [\alpha_2, \beta_2]\}$.
Then x, y ∈ $A_{(\alpha, \beta)}$
$\Rightarrow x\alpha z\beta y \in A_{(\alpha, \beta)}$
$\Rightarrow V_A(x\alpha z\beta y) \geq \min\{V_A(x), V_A(y)\}$
Hence A is a vague bi-ideal of R.

Theorem 3.6: Let S be a non-empty subset of R. Then the characteristic set of S, δ_S is a vague bi-ideal of R if and only if S is a bi-ideal of R.

Proof: Suppose δ_S is a vague bi-ideal of R.
From theorem:3.9[1], S is a sub Γ-semiring of R
Let x, y ∈ S, z ∈ R and α, β ∈ Γ.
We have $V_{\delta_S}(x\alpha z\beta y) \geq \min\{V_{\delta_S}(x), V_{\delta_S}(y)\} = [1,1]$.
which implies that xαzβy ∈ S.
Hence S is a bi-ideal of R.
Conversely assume that S is a bi-ideal of R.
From theorem:3.9[1], δ_S is a vague Γ-semiring of R.
Let x, y, z ∈ R and α, β ∈ Γ.
If x, y ∈ S, then xαzβy ∈ S.
So, $V_{\delta_S}(x\alpha z\beta y) = [1,1] = \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$.
If x, y ∉ S, then $V_A(x) = V_A(y) = [0, 0]$.
So, $V_{\delta_S}(x\alpha z\beta y) \geq \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$.
If x ∉ S and y ∈ S, then $V_A(x) = [0, 0]$ and $V_A(y) = [1,1]$.
So, $V_{\delta_S}(x\alpha z\beta y) \geq \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$.
A Similar argument for x ∈ S and y ∉ S.
Hence δ_S is a vague bi-ideal of R.

Theorem 3.7: If R is regular, then every vague bi-ideal of R is a vague Γ-semiring of R.

Proof: Let A = (t_A, f_A) be a vague bi-ideal of R.
Let x, y ∈ R and γ ∈ Γ.
Then $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$

Since R is regular, we have $y = y\alpha z\beta y$, where $z \in R$ and $\alpha, \beta \in \Gamma$.
Now, $V_A(x\gamma y) = V_A(x\gamma(y\alpha z\beta y)) = V_A(x\gamma(y\alpha z)\beta y) \geq \min\{V_A(x), V_A(y)\}$.
Hence A is a vague Γ-semiring of R.

Theorem 3.8: Let R be regular and let $A = (t_A, f_A)$ be a vague set of R. Then A is a vague bi-ideal of R if and only if A is a (1,2)-ideal of R.

Proof: Suppose A is a vague bi-ideal of R.
Let $x, w, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$.
Now, $V_A(x\alpha w\beta(y\gamma z)) = V_A((x\alpha w\beta y)\gamma z)$
$\geq \min\{ V_A(x\alpha w\beta y), V_A(z)\}$
$\geq \min\{ V_A(x), V_A(y), V_A(z)\}$.
Thus A is a (1,2)-ideal of R.
Conversely suppose that A is a (1,2)-ideal of R.
Let $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.
Since R is regular, we have $x\alpha y \in x\Gamma R\Gamma x$, that implies $x\alpha y = x\gamma s\delta x$, for some $s \in R$ and $\gamma, \delta \in \Gamma$.
Now, $V_A(x\alpha y\beta z) = V_A((x\gamma s\delta x)\beta z)$
$= V_A(x\gamma s\delta(x\beta z))$
$\geq \min\{V_A(x), V_A(x), V_A(z)\}$
$= \min\{V_A(x), V_A(z)\}$.
Hence A is a vague bi-ideal of R.

Theorem 3.9: Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be vague bi-ideals of R. Then $A\cap B$ is also a vague bi-ideal of R.

Proof: From theorem:3.13[1], $A\cap B$ is a vague Γ-semiring R.
Let $x, y, z \in R$; $\alpha, \beta \in \Gamma$.
Now, $V_{A\cap B}(x\alpha y\beta z) = \min\{V_A(x\alpha y\beta z), V_B(x\alpha y\beta z)\}$
$\geq \min\{\min\{V_A(x), V_A(z)\}, \min\{V_B(x), V_B(z)\}\}$
$= \min\{\min\{V_A(x), V_A(z), V_B(x), V_B(z)\}\}$
$= \min\{\min\{V_A(x), V_B(x)\}, \min\{V_A(z), V_B(z)\}\}$
$= \min\{V_{A\cap B}(x), V_{A\cap B}(z)\}$
Hence $A\cap B$ is also a vague bi-ideal of R.

Theorem 3.10: : Let f be a homomorphism from a Γ_1-semiring R_1 onto a Γ_2-semiring R_2. Let B be a vague bi-ideal of R_2. Then the pre-image of B, $f^{-1}(B)$ is a vague bi-ideal of R_1.

Proof: From theorem:3.1[3], $f^{-1}(B)$ is a vague Γ_1-semiring of R_1.
Let $x, y, z \in R_1$; $\alpha, \beta \in \Gamma_1$.
Now, $V_{f^{-1}(B)}(x\alpha y\beta z) = V_B(f(x\alpha y\beta z))$
$= V_B(f(x)g(\alpha)f(y)g(\beta)f(z))$
$\geq \min\{V_B(f(x)), V_B(f(z))\}$
$= \min\{V_{f^{-1}(B)}(x), V_{f^{-1}(B)}(z)\}$
Hence $f^{-1}(B)$ is a vague bi-ideal of R_1.

Theorem 3.11: Let f be an epimorphism from a Γ_1-semiring R_1 onto a Γ_2-semiring R_2. Let A be a vague bi-ideal of R_1. Then the homomorphic image of A, f(A) is a vague bi-ideal of R_2.

Proof: Let $x, y, z \in R_2$; $\alpha, \beta \in \Gamma_2$.
If either $f^{-1}(x)$ or $f^{-1}(y)$ is empty then the result is trivially satisfied.
Suppose neither $f^{-1}(x)$ nor $f^{-1}(y)$ is non-empty.
Let $x_0 \in f^{-1}(x)$ and $y_0 \in f^{-1}(y)$ be such that $V_A(x_0) = \sup V_A(a)$ where $a \in f^{-1}(x)$ and $V_A(y_0) = \sup V_A(b)$ where $b \in f^{-1}(y)$.
From theorem:3.2[3], f(A) is a vague Γ_2-semiring of R_2.
Now, $V_{f(A)}(x\alpha y\beta z) = \sup_{z\in f^{-1}(x\alpha y\beta z)} V_A(z)$
$\geq V_A(z), z \in f^{-1}(x\alpha y\beta z)$
$= V_A(x_0\alpha_1 y_0\beta_1 z_0), \alpha_1, \beta_1 \in \Gamma_1$
$\geq \min\{V_A(x_0), V_A(z_0)\}$
$= \min\{V_{f(A)}(x), V_{f(A)}(z)\}$.
Hence f(A) is a vague bi-ideal of R_2.

Theorem 3.12: Let $A = (t_A, f_A)$ be a vague set of R. If A is a vague bi-ideal of R then $A\Gamma A \subseteq A$ and $A\Gamma B\Gamma A \subseteq A$, where B is a vague characteristic set of R.

Proof: Suppose A is a vague bi-ideal of R.
Let $x \in R$.
Now, $V_{A\Gamma A}(x) = \sup\{\min\{V_A(y), V_A(z)\}/ x = y\gamma z$, where $y, z \in R; \gamma \in \Gamma\} \leq V_A(y\gamma z) = V_A(x)$.
Thus $A\Gamma A \subseteq A$.
Also,
$V_{A\Gamma B\Gamma A}(x) = \sup\{\min\{V_{A\Gamma B}(y), V_B(z)\}/ x = y\gamma z\}$
$= \sup\{\min\{\sup\{\min\{V_A(p), V_B(q)/ y = p\alpha q\}\}, V_B(z)\}/ x = y\gamma z\}$
$= \sup\{\min\{V_A(p), V_B(z)\}/ x = p\alpha q\gamma z\}$
$\leq V_A(p\alpha q\gamma z)$
$= V_A(x)$.
Thus $A\Gamma B\Gamma A \subseteq A$.

Theorem 3.13: Suppose R is regular. Then the following conditions are equivalent:
1. Every bi-ideal of R is a left(right) ideal of R.
2. Every vague bi-ideal of R is a left(right) vague ideal of R.

Proof: Suppose every bi-ideal of R is a left ideal of R.
Let $A = (t_A, f_A)$ be a vague bi-ideal of R.
Let $x, y \in R$ and $\gamma \in \Gamma$.
We have $y\Gamma R\Gamma y$ is a bi-ideal of R.
By our assumption, $y\Gamma R\Gamma y$ is a left ideal of R.
Now, $x\gamma y \in R\Gamma(y\Gamma R\Gamma y) \subseteq y\Gamma R\Gamma y$.
So, $x\gamma y = y\alpha z\beta y$, where $z\in R$ and $\alpha, \beta \in \Gamma$.
Now, $V_A(x\gamma y) = V_A(y\alpha z\beta y) \geq \min\{V_A(y), V_A(y)\} = V_A(y)$.
Hence A is a left vague ideal of R.
Conversely suppose every vague bi-ideal of R is a left vague ideal of R.
Let I be a bi-ideal of R.
By theorem:3.6, vague characteristic set of I is a vague bi-ideal of R.
By our assumption, vague characteristic set of I is a left vague ideal of R.
By theorem:3.10[2], I is a left ideal of R.
Hence every bi-ideal of R is a left ideal of R.
Similarly we can prove right ideals also.

4. Vague Quasi Ideal of a Γ-Semiring

In this section we introduce the concept of vague quasi ideal of R and we proved intersection of a right vague ideal and a left vague ideal of R is a vague quasi ideal of R and we characterize regular Γ-semiring in terms of vague bi-ideals and vague quasi ideals.

Definition 4.1: A vague set $A = (t_A, f_A)$ of R is said to be a vague quasi ideal of R if for all $x, y \in R$; $\gamma \in \Gamma$,

1. $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and
2. $(A \Gamma \chi) \cap (\chi \Gamma A) \subseteq A$.

Example 4.2: Let R be the set of natural numbers with zero and Γ be the set of positive even integers. Then R, Γ are additive commutative semigroups.
Define the mapping $R \times \Gamma \times R \to R$ by $a\alpha b$ usual product of a, α, b, $\forall\, a, b \in R$; $\alpha \in \Gamma$.
Then R is a Γ-semiring.
Let $A = (t_A, f_A)$, where $t_A : R \to [0, 1]$ and $f_A : R \to [0, 1]$ defined by

$$t_A(x) = \begin{cases} 0.8, \text{if } x = 0 \\ 0.5, \text{if } x \text{ is positive} \\ 0.4, \text{if } x \text{ is negative} \end{cases} \quad \text{and}$$

$$f_A(x) = \begin{cases} 0.1, \text{if } x = 0 \\ 0.3, \text{if } x \text{ is positive} \\ 0.5, \text{if } x \text{ is negative} \end{cases}$$

Then A is a vague quasi ideal of R.

Theorem 4.3: A vague set $A = (t_A, f_A)$ of R is a left(right) vague ideal of R if and only if for all $x, y \in R$,
(i). $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$
(ii). $\chi \Gamma A \subseteq A$ $(A\Gamma\chi \subseteq A)$.

Proof: Suppose that A is a left vague ideal of R.
Then (i) is satisfied.
Let $x \in R$.
$V_{\chi\Gamma A}(x) = \sup\{\min\{V_\chi(y), V_A(z)\}/\ x= y\gamma z,\ y, z \in R; \gamma \in \Gamma\} = V_A(z) \leq V_A(y\gamma z) = V_A(x)$.
Thus $\chi \Gamma A \subseteq A$.
Conversely suppose that (i) and (ii) are satisfied.
Let $x, y \in R$; $\gamma \in \Gamma$.
$V_A(x\gamma y) \geq V_{\chi\Gamma A}(x\gamma y) = \sup\{\min\{V_\chi(x), V_A(y)\}\} = V_A(y)$.
Hence A is a left vague ideal of R.
Similarly we can prove other case also.

Theorem 4.4: Any left(right) vague ideal is a vague quasi ideal and any vague quasi ideal is a vague bi-ideal of R.

Proof: Let $A = (t_A, f_A)$ be a left(right) vague ideal of R.
Then from theorem:4.3, for all $x, y \in R$, (i). $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$ and (ii). $\chi \Gamma A \subseteq A$ $(A\Gamma\chi \subseteq A)$.
That implies $(A \Gamma \chi) \cap (\chi \Gamma A) \subseteq \chi \Gamma A (A\Gamma\chi \subseteq A) \subseteq A$.
Hence A is a vague quasi ideal of R.
Let A be a vague quasi ideal of R.
Let $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.
Since A is a vague quasi ideal of R, we have $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$.
Also, $V_A(x\alpha y) \geq V_{(A \Gamma \chi) \cap (\chi \Gamma A)}(x\alpha y)$
$= \min\{V_{A \Gamma \chi}(x\alpha y), V_{\chi \Gamma A}(x\alpha y)\}$
$= \min\{\sup\{\min\{V_A(x), V_\chi(y)\}, \sup\{\min\{V_\chi(x), V_A(y)\}\}$
$= \min\{V_A(x), V_A(y)\}$
Thus A is a vague Γ-semiring of R.
Now, $V_A(x\alpha y\beta z) \geq V_{(A \Gamma \chi) \cap (\chi \Gamma A)}(x\alpha y\beta z)$
$= \min\{V_{A \Gamma \chi}(x\alpha y\beta z), V_{\chi \Gamma A}(x\alpha y\beta z)\}$
$= \min\{\sup\{\min\{V_A(x), V_\chi(y\beta z)\}, \sup\{\min\{V_\chi(x\alpha y), V_A(z)\}\}$
$= \min\{V_A(x), V_A(z)\}$
Therefore $V_A(x\alpha y\beta z) \geq \min\{V_A(x), V_A(z)\}$.
Hence A is a vague bi-ideal of R.

Theorem 4.5: If R is regular, then every vague bi-ideal of R is a vague quasi ideal of R.

Proof: Suppose $A = (t_A, f_A)$ is a vague bi-ideal of R.
Let $x, y \in R$.
Then $V_A(x + y) \geq \min\{V_A(x), V_A(y)\}$.
We have to prove that $(A\Gamma\chi)\cap(\chi\Gamma A) \subseteq A$.
Case (i). Suppose $V_{A\Gamma\chi}(x) \leq V_A(x)$, then
$V_{(A\Gamma\chi)\cap(\chi\Gamma A)}(x) = \min\{V_{A\Gamma\chi}(x), V_{\chi\Gamma A}(x)\} \leq V_{A\Gamma\chi}(x) \leq V_A(x)$.
So $(A\Gamma\chi)\cap(\chi\Gamma A) \subseteq A$.
Hence A is a vague quasi ideal of R.
Case (ii). Suppose $V_{A\Gamma\chi}(x) > V_A(x)$, then
$V_A(x) < V_{A\Gamma\chi}(x) = \sup\{\min\{V_A(a), V_\chi(b)\}/\ x= a\alpha b\} = V_A(a)$.
i.e., $V_A(x) < V_A(a)$.
Now, $V_{\chi\Gamma A}(x) = \sup\{\min\{V_\chi(c), V_A(d)\}/\ x= c\beta d\} = V_A(d)$.
Since R is regular, there exists $s \in R$ and $\gamma, \delta \in \Gamma$ such that $x = x\gamma s\delta x$.
Since A is vague bi-ideal of R, we have
$V_A(x) = V_A(x\gamma s\delta x) = V_A((a\alpha b)\ \gamma s\delta(c\beta d)) = V_A(a\alpha(b\gamma s\delta c)\beta d) \geq \min\{V_A(a), V_A(d)\}$.
If $\min\{V_A(a), V_A(d)\} = V_A(a)$, then $V_A(x) \geq V_A(a)$.
Which is a contradiction.
So, $\min\{V_A(a), V_A(d)\} = V_A(d)$.
i.e., $V_A(x) \geq V_A(d)$.
Now, $V_{(A\Gamma\chi)\cap(\chi\Gamma A)}(x) = \min\{V_{A\Gamma\chi}(x), V_{\chi\Gamma A}(x)\} \leq V_{\chi\Gamma A}(x) \leq V_A(d) \leq V_A(x)$.
Therefore $(A\Gamma\chi)\cap(\chi\Gamma A) \subseteq A$.
Hence A is a vague quasi ideal of R.

Theorem 4.6: Intersection of a right vague ideal and a left vague ideal of R is a vague quasi ideal of R.

Proof: Let $A =(t_A, f_A)$ be a right vague ideal of R and $B = (t_B, f_B)$ be a left vague ideal of R.
Let $x, y \in R$.
Then $V_{A\cap B}(x + y) \geq \min\{V_{A\cap B}(x), V_{A\cap B}(y)\}$.
Since A is a right vague ideal and B is a left vague ideal of R, $A \Gamma \chi \subseteq A$ and $\chi \Gamma B \subseteq B$.
Therefore $((A\cap B) \Gamma \chi)\cap(\chi \Gamma(A\cap B)) \subseteq (A \Gamma \chi)\cap(B \Gamma \chi) \subseteq A\cap B$.
Hence $A\cap B$ is a vague quasi ideal of R.

Theorem 4.7: Let S be a non-empty subset R. Then vague characteristic set of S, δ_S is a vague quasi ideal of R if and only if S is a quasi ideal of R.

Proof: Suppose δ_S is a vague quasi ideal of R.
Let $x, y \in S$.
We have $V_{\delta_S}(x + y) \geq \min\{V_{\delta_S}(x), V_{\delta_S}(y)\} = [1, 1]$.
which implies that $x + y \in S$.

Let $x \in (S\Gamma R)\cap(R\Gamma S)$.
$\Rightarrow x \in S\Gamma R$ and $x \in R\Gamma S$.
$\Rightarrow x = a\alpha b = c\beta d$, where $a, d \in S$; $b, c \in R$ and $\alpha, \beta \in \Gamma$.
We have $V_{\delta_S}(x) \geq V_{(\delta_S\Gamma\chi)\cap(\chi\Gamma\delta_S)}(x)$
$= \min\{V_{\delta_S\Gamma\chi}(x), V_{\chi\Gamma\delta_S}(x)\}$
$= \min\{\sup\{\min\{V_{\delta_S}(a), V_{\chi}(b)\}\}, \sup\{\min\{V_{\chi}(c), V_{\delta_S}(d)\}\}\}$
$= [1, 1]$.
Therefore $x \in S$.
Hence S is a quasi ideal of R.
Conversely suppose that S is a quasi ideal of R.
Let $x, y \in R$.
If $x, y \in S$, then $x + y \in S$ and $x\gamma y \in S$.
So, $V_{\delta_S}(x + y) = [1,1] = \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$
If $x, y \notin S$, then $V_A(x) = [0,0] = V_A(y)$.
So, $V_{\delta_S}(x + y) = [0,0] = \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$
If $x \notin S$ and $y \in S$, then $V_A(x) = [0,0]$ and $V_A(y) = [1, 1]$.
So, $V_{\delta_S}(x + y) = [0,0] = \min\{V_{\delta_S}(x), V_{\delta_S}(y)\}$
A Similar argument for $x \in S$ and $y \notin S$.
Now, let $x \in R$.
If $x \in S$, then $V_{\delta_S}(x) = [1, 1]$.
We have $V_{(\delta_S\Gamma\chi)\cap(\chi\Gamma\delta_S)}(x) \leq [1, 1] = V_{\delta_S}(x)$.
Suppose $x \notin S$, then $V_{\delta_S}(x) = [0,0]$.
Now, $x \notin S$ implies $x \notin (R\Gamma S)\cap(S\Gamma R)$.
Then 3 cases arise.

Case (i). $x \notin R\Gamma S$ and $x \notin S\Gamma R$.
Case (ii). $x \in R\Gamma S$ and $x \notin S\Gamma R$.
Case (iii). $x \notin R\Gamma S$ and $x \in S\Gamma R$.

Case (i). $x \notin R\Gamma S$ and $x \notin S\Gamma R$.
Then $x = a\alpha b$ and $x = c\beta d$, where $b, c \notin S$; $a, d \in R$ and $\alpha, \beta \in \Gamma$.
Now, $V_{(\delta_S\Gamma\chi)\cap(\chi\Gamma\delta_S)}(x) = \min\{V_{\delta_S\Gamma\chi}(x), V_{\chi\Gamma\delta_S}(x)\}$
$= \min\{\sup\{\min\{V_{\delta_S}(c), V_{\chi}(d)\}\}, \sup\{\min\{V_{\chi}(a), V_{\delta_S}(b)\}\}\}$
$= [0, 0]$
$= V_{\delta_S}(x)$.
Case (ii). $x \in R\Gamma S$ and $x \notin S\Gamma R$.
Then $x = a\alpha b$ and $x = c\beta d$, where $c \notin S$; $b \in S$; $a, d \in R$ and $\alpha, \beta \in \Gamma$.
Now, $V_{(\delta_S\Gamma\chi)\cap(\chi\Gamma\delta_S)}(x) = \min\{V_{\delta_S\Gamma\chi}(x), V_{\chi\Gamma\delta_S}(x)\}$
$= \min\{\sup\{\min\{V_{\delta_S}(c), V_{\chi}(d)\}\}, \sup\{\min\{V_{\chi}(a), V_{\delta_S}(b)\}\}\}$
$= [0, 0]$
$= V_{\delta_S}(x)$.
Case (iii). $x \notin R\Gamma S$ and $x \in S\Gamma R$.
Similar to case(ii).
Therefore in any case $(\delta_S\Gamma\chi)\cap(\chi\Gamma\delta_S) \subseteq \delta_S$.
Hence δ_S is a vague quasi ideal of R.

Theorem 4.8: R is regular and intra-regular if and only if every quasi ideal of R is idempotent.

Proof: Suppose R is regular and intra-regular.
Let S be a quasi ideal of R.
Then $(R\Gamma S)\cap(S\Gamma R) \subseteq S$.
Therefore $S\Gamma S \subseteq S$.
Let $x \in S$.
Since R is regular and intra-regular, there exists $a, p, q \in R$ and $\alpha, \beta, \gamma, \delta, \eta \in \Gamma$ such that $x = x\alpha a\beta x$ and $x = p\gamma x\delta x\eta q$.
So, $x = x\alpha a\beta x$
$= x\alpha a\beta x\alpha a\beta x$
$= x\alpha a\beta(p\gamma x\delta x\eta q)\alpha a\beta x$
$= (x\alpha a\beta p\gamma x)\delta(x\eta q\alpha a\beta x) \in S\Gamma S$.
$\Rightarrow S \subseteq S\Gamma S$.
Therefore $S = S\Gamma S$.
Hence every quasi ideal of R is idempotent.
Conversely suppose that every quasi ideal of R is idempotent.
Let $x \in R$.
Consider the quasi ideal $S[x] = \{x\}\cup(x\Gamma R \cap R\Gamma x)$ generated by x of R.
By our assumption $S[x] = S[x]\,\Gamma\,S[x]$.
So, $x \in S[x]\Gamma S[x] = (x\Gamma x) \cup (x\Gamma(x\Gamma R \cap R\Gamma x)) \cup ((x\Gamma R \cap R\Gamma x)\Gamma x) \cup ((x\Gamma R \cap R\Gamma x)\Gamma(x\Gamma R \cap R\Gamma x))$.
That implies $x = x\alpha a\beta x$ and $x = p\gamma x\delta x\eta q$, for some $a, p, q \in$ Rand $\alpha, \beta, \gamma, \delta, \eta \in \Gamma$.
Hence R is regular and intra-regular.

Lemma 4.9: Suppose A and B are non empty subsets of R. Then $\chi_A\Gamma\chi_B = \chi_{A\Gamma B}$, where χ_A is the characteristic set of A and χ_B is the characteristic set of B.

Theorem 4.10: Every quasi ideal of R is idempotent if and only if every vague quasi ideal of R is idempotent.

Proof: Every quasi ideal of R is idempotent.
Let $A = (t_A, f_A)$ be a vague quasi ideal of R.
We have $A\Gamma A \subseteq (A\,\Gamma\chi)\cap(\chi\,\Gamma A) \subseteq A$.
From theorem:4.8, R is regular and intra-regular.
Let $x \in R$.
Therefore $x = x\alpha a\beta x$ and $x = p\gamma x\delta x\eta q$, for some $a, p, q \in R$ and $\alpha, \beta, \gamma, \delta, \eta \in \Gamma$.
Now, $x = x\alpha a\beta x = x\alpha a\beta(x\alpha a\beta x) = x\alpha a\beta(\,p\gamma x\delta x\eta q)\alpha a\beta x = (x\alpha a\beta p\gamma x)\delta(x\eta q\alpha a\beta x)$.
Since every vague quasi ideal is a vague bi ideal of R, we have
$V_{A\Gamma A}(x) = V_{A\Gamma A}((x\alpha a\beta p\gamma x)\delta(x\eta q\alpha a\beta x))$
$=\sup\{\min\{V_A(x\alpha a\beta p\gamma x), V_A(x\eta q\alpha a\beta x)\}\}$
$\geq \min\{V_A(x\alpha a\beta p\gamma x), V_A(x\eta q\alpha a\beta x)\}$
$= \min\{\min\{V_A(x), V_A(x)\}, \min\{V_A(x), V_A(x)\}\}$
$= V_A(x)$
That implies $A \subseteq A\Gamma A$.
Therefore $A = A\Gamma A$.
Hence every vague quasi ideal of R is idempotent.
Conversely suppose every vague quasi ideal of R is idempotent.
Let S be a quasi ideal of R.
By theorem:4.7, δ_S is a vague quasi ideal of R.
Therefore $\delta_S = \delta_S\Gamma\delta_S = \delta_{S\Gamma S}$.

that implies $S\Gamma S = S$.
Hence every quasi ideal of R is idempotent.

Theorem 4.11: The following are equivalent.
1.R is regular and intra-regular.
2. Every vague quasi ideal of R is idempotent.
3. Every vague bi-ideal of R is idempotent.

Proof: (1) $\Rightarrow$ (3)
Suppose R is regular and intra-regular.
Let $A = (t_A, f_A)$ be a vague bi-ideal of R.
Let $x \in R$.
Then $x = x\alpha a\beta x$ and $x = p\gamma x\delta x\eta q$, for some $a, p, q \in R$ and $\alpha, \beta, \gamma, \delta, \eta \in \Gamma$.
Now, $V_{A\Gamma A}(x) = V_{A\Gamma A}((x\alpha a\beta x)$
$= V_{A\Gamma A}(x\alpha a\beta x\alpha a\beta x)$
$= V_{A\Gamma A}((x\alpha a\beta p\gamma x)\delta(x\eta q\alpha a\beta x))$
$= \sup\{\min\{V_A(x\alpha a\beta p\gamma x), V_A(x\eta q\alpha a\beta x)\}\}$
$\geq \min\{V_A(x\alpha a\beta p\gamma x), V_A(x\eta q\alpha a\beta x)\}$
$= \min\{\min\{V_A(x),V_A(x)\}, \min\{ V_A(x),V_A(x)\}\}$
$= V_A(x)$
Therefore $A \subseteq A\Gamma A$.
Also, $V_{A\Gamma A}(x) = \sup\{\min\{V_A(y), V_A(z)\}/ x = y\gamma z\} \leq V_A(y\gamma z)$
$= V_A(x)$.
Therefore $A = A\Gamma A$.
Hence every vague bi-ideal of R is idempotent.
(3) $\Rightarrow$ (2)
Suppose every vague bi-ideal of R is idempotent.
Since every vague quasi ideal of R is vague bi-ideal of R, we have every vague quasi ideal of R is idempotent.
(2) $\Rightarrow$(1)
Suppose every vague quasi ideal of R is idempotent.
By theorem:4.8 and 4.10, R is regular and intra-regular.
Hence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

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References

[1] Bhargavi.Y and Eswarlal.T, “Vague Γ-semirings”, Global Journal of Pure and Applied Mathematics, Vol. 11, No. 1, pp.117-127, 2015.
[2] Bhargavi.Y and Eswarlal.T, “Vague Ideals and Normal Vague Ideals in Γ-semirings”, International Journal of Innovative Research and Development, Vol. 4, Issue. 3, pp.1-8, 2015.
[3] Bhargavi.Y and Eswarlal.T, “Vague Homomorphism of a Γ-semiring”, accepted for publication in International Journal of Scientific and Innovative Mathematical Research.
[4] Gau.W.L and Buehrer.D.J, “Vague Sets”, IEEE Transactions on systems, man and cybernetics, Vol. 23, No.2, pp.610-613, 1993.
[5] Good R.A., Hughes D.R., Associated groups for a semigroup. Bull. Amer. Math. Soc., 58, pp.624-625, 1952.
[6] John N Mordeson and Malik.D.S, “Fuzzy Commutative Algebra”, World Scientific Publishing Co. Pvt. Ltd, 1998.
[7] Lajos.S and Szasz.F, On the bi-ideals in Associative ring. Proc. Japan Acad., 46, pp.505-507, 1970.
[8] Mandal.D, "Fuzzy bi-ideals and Fuzzy Quasi Ideals in Ordered Semirings", Gulf Journal of Mathematics, Vol. 2, Issue 4, pp.60-67, 2014.
[9] M.K.Rao, “Γ-semiring1”, Southeast Asian Bulletin of Mathematics, 19, pp.49-54, 1995.
[10] Nobusawa.N, “On generalization of the ring theory ”, Osaka J. Math. 1, pp.185-190, 1978.
[11] Sardar.S.K, Majumder.S.K, "On Fuzzy Ideals in Γ-semigroups", International Journal of Algebra, Vol.3, No. 16, pp.775-784, 2009.
[12] Zadeh.L.A, “Fuzzy sets”, Information and Control 8, pp.338-353, 1965.