Poisson-Gamma Counting Process as a Discrete Survival Model

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Abstract: ***In this paper a Poisson–Gamma distribution has been proposed, which is obtained by compounding a Poisson distribution with a two parameter Gamma distribution. Here the pmf of the proposed distribution (PGD) is derived. The expressions for raw moments, central moments, coefficients of skewness and kurtosis have been derived. Survival and Hazard functions of proposed distribution are also obtained. The estimator of the parameters have been obtained by method of Moments as well as method of Maximum Likelihood. The proposed distribution has found to be a good fit of Kemp & Kemp survival data (1965).***

Keywords: Poisson–Gamma distribution, Negative Binomial Distribution, Maximum Likelihood Estimator, Method of Moments, Survival Function, Hazard Function etc.

1. Introduction

In reliability/survival lifetime modeling, it is common to treat failure data as being continuous, implying some degree of precision in measurement. Too often in practice, however, failures are either noted at regular inspection intervals, occurs in a discrete process or are simply recorded in bins. In life testing experiments or survival time data, it is sometimes impossible or inconvenient to measure the life length on a continuous scale. Thus, it is essential to construct discrete lifetime models for discrete failure survival data, Lai (2013). Roy, D. (2004) have also discussed the properties of discrete Rayleigh distribution. Finite range discrete lifetime distributions are discussed by Lai, C.D. et al. (1995).

Let the random variable X has a Poisson distribution with pmf as

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; x = 0, 1, 2, \dots \; ; \quad \lambda > 0 \qquad (1)$$

Let us consider a two parameter Gamma distribution with pdf is given by

$$h(x;\alpha,\beta) = \frac{\alpha^{\beta}}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1} \; ; \alpha, \beta > 0; \quad 0 < x < \infty \qquad (2)$$

Now if the parameter λ of the above Poisson distribution (Equation-1) is distributed as the above Gamma distribution (Equation-2). The resultant pmf of X, g(x; α, β) may be obtained as

$$g(x;\alpha,\beta) = \int_0^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} \frac{\alpha^{\beta}}{\Gamma(\beta)} e^{-\alpha\lambda} \lambda^{\beta-1} \, d\lambda \qquad (3)$$

or,

$$g(x;\alpha,\beta) = \frac{\Gamma(x+\beta)}{x!\,\Gamma(\beta)} \left(\frac{\alpha}{1+\alpha}\right)^{\beta} \left(\frac{1}{1+\alpha}\right)^{x}; x = 0,1,2,\dots; \alpha, \beta > 0 \qquad (4)$$

which is the pmf of Poisson-Gamma distribution (Counting Process). Withers and Nadarjah (2011) has discussed some properties of Poisson-Gamma distribution.

Equation (4) may be expressed as

$$g(x;\alpha,\beta) = \frac{(x+\beta-1)!}{x!(\beta-1)!} \left(\frac{\alpha}{1+\alpha}\right)^{\beta} \left(\frac{1}{1+\alpha}\right)^{x} \; ;$$

$$x = 0,1,2,\dots \; ; \; \alpha > 0, \beta = 1,2,\dots \qquad (5)$$

which is the pmf of classical Negative Binomial Distribution (as generated by the number of independent trials necessary to obtain β occurrences of an event which has constants probability $p = \left(\frac{\alpha}{1+\alpha}\right)$ of occurrence at each trial, Johnson and Kotz (1969)).

$$\text{or,} \quad g(x;\alpha,\beta) = \binom{x+\beta-1}{\beta-1} \alpha^{\beta} (1+\alpha)^{-(x+\beta)} \; ;$$

$$x = 0,1,2,\dots \; ; \alpha > 0, \beta = 1,2,.. \qquad (6)$$

If β is not a non-negative integer, Equation-4 may be termed as **'Psuedo'** Negative Binomial Distribution.

Thus, this hierarchical proposed distribution (Equation-4) may be named as a **'Generalized'** Negative Binomial Distribution in the sense that β is either non-negative integer or a non-negative real number. Here we get

$$\sum_{x=1}^{\infty} g(x;\alpha,\beta) = 1 \qquad (7)$$

Students (1907) used the Negative Binomial Distribution as an alternative to the Poisson distribution in describing counts on the plates of haemacytometer. The Negative Binomial Distribution was studied by Fisher (1941), Jeffreys (1941) and Anscombe (1950) under different parameterization. It has been shown to be the limiting form of Eggenberg and Polya's urn model by Patil et al. (1984) and Gamma mixture of Poisson distribution by Greenwood and Yule (1920), addition of a set of correlated Poisson distributions by Martiz (1952). The Negative Binomial Distribution also arises out of a few stochastic processes as pointed by McKendrick (1914), Irwin (1941), Lundeberg (1940) and Kendall (1949). This distribution, being more flexible than Poison distribution, enjoys a plethora of applications. It can be used to model accident data, psychological data, economics data, consumer data, medical data, defense data and so on. Chandra and Roy (2012) proposed a continuous version of the Negative Binomial Distribution by considering a particular type of survival function.

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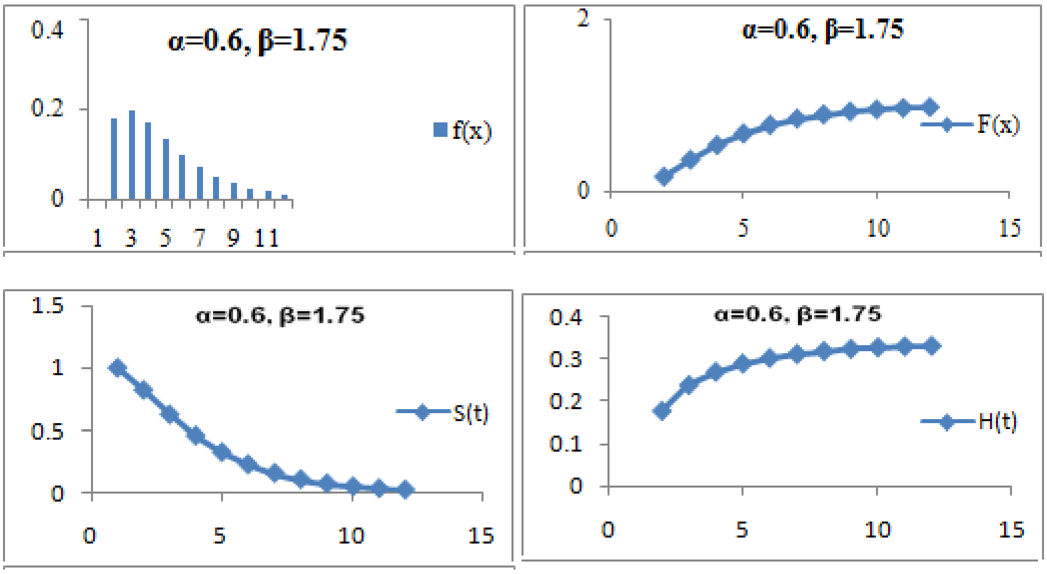

Figure 1: Showing of pmf, cdf, survival function, hazard function for PGD (Equation-4)

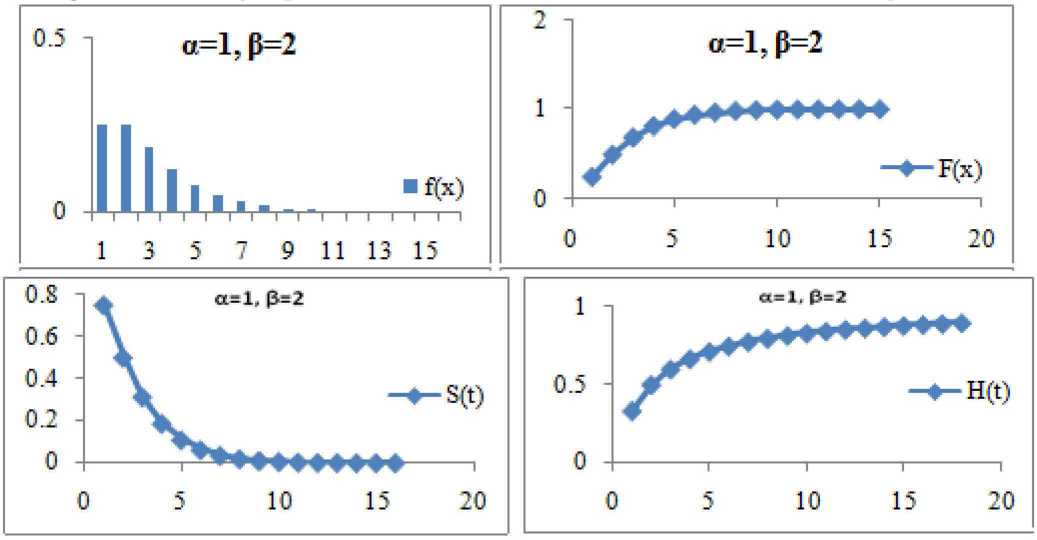

Figure 2: Showing of pmf, cdf, survival function, hazard function for PGD (Equation-6)

2. Moments of Poisson-Gamma Distribution (PGD)(Equation-6)

The first four raw moments about origin and their corresponding central moments of Poisson Gamma Distribution (as Negative Binomial Distribution) are

$$\mu_1{}' = \frac{\beta}{\alpha}$$

$$\mu_2{}' = \frac{(\beta)(\alpha+\beta+1)}{\alpha^2}$$

$$\mu_3{}' = \frac{(\beta)(\alpha^2+\beta^2+3\alpha+3\beta+3\alpha\beta+2)}{\alpha^3}$$

$$\mu_4{}' = \frac{(\beta)(\alpha^3+\beta^3+7\alpha^2\beta+6\alpha\beta^2+6\beta^2+18\alpha\beta+11\beta+6)}{\alpha^4}$$

and their corresponding central moments are

$$\mu_1 = \mu_1{}' = \frac{\beta}{\alpha}$$

$$\mu_2 = \frac{(\beta)(1+\alpha)}{\alpha^2}$$

$$\mu_3 = \frac{(\beta)(1+\alpha)(2+\alpha)}{\alpha^3}$$

$$\mu_4 = \frac{(\beta)(1+\alpha)[\alpha^2+3(2+\beta)(1+\alpha)]}{\alpha^4}$$

Thus, the mean, variance, skewness, kurtosis and their coefficients are given by

$$\text{Mean }(\mu_1) = \mu_1{}' = \frac{\beta}{\alpha} \quad (8)$$

$$\text{Variance }(\mu_2) = \frac{(\beta)(1+\alpha)}{\alpha^2} \quad (9)$$

$$\beta_1 = \frac{\mu_3{}^2}{\mu_2{}^3} = \frac{(2+\alpha)^2}{(\beta)(1+\alpha)} \quad (10)$$

$$\beta_2 = \frac{\mu_4}{\mu_2{}^2} = \frac{\alpha^2+3(2+\beta)(1+\alpha)}{(\beta)(1+\alpha)} \quad (11)$$

$$\gamma_1 = \sqrt{\beta_1} = (2+\alpha)\sqrt{\frac{1}{(\beta)(1+\alpha)}} \quad (12)$$

$$\gamma_2 = \beta_2 - 3 = \frac{\alpha^2+6\alpha+6}{(\beta)(1+\alpha)} \quad (13)$$

3. Mode of Poisson-Gamma Distribution (PGD)

$$\frac{P(x)}{P(x-1)} = \frac{x+\beta-1}{x(1+\alpha)} \quad (14)$$

Now, we can discuss the following cases:

Case I: When $\left(\frac{\beta-1}{\alpha}\right)$ is not an integer

Let us suppose that S is the integral part of $\frac{\beta-1}{\alpha}$

So that

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$\frac{P(1)}{P(0)} > 1, \frac{P(2)}{P(1)} > 1, \ldots, \frac{P(S-1)}{P(S)} > 1, \frac{P(S)}{P(S-1)} > 1$ and $\frac{P(S+1)}{P(S)} < 1, \frac{P(S+2)}{P(S+1)} < 1, \ldots$ (15)

P(S) is the maximum value, in this case this distribution is unimodal and the integral part of $\left(\frac{\beta-1}{\alpha}\right)$ is the unique modal value.

Case II: When $\left(\frac{\beta-1}{\alpha}\right)$ = k (say) is an integer

Here as in case I, we have

$\frac{P(1)}{P(0)} > 1, \frac{P(2)}{P(1)} > 1, \ldots, \frac{P(k-1)}{P(k-2)} > 1, \frac{P(k)}{P(k-1)} = 1$ and $\frac{P(k+1)}{P(k)} < 1, \frac{P(k+2)}{P(k+1)} < 1, \ldots$ (16)

In this case the distribution is bimodal and two modes are at (k-1) and k such that $\left(\frac{\beta-1}{\alpha} - 1\right)$ and $\left(\frac{\beta-1}{\alpha}\right)$

Notes 1: If $0 < \beta \le 1$, the mode always lies at zero

2: If $\left(\frac{\beta-1}{\alpha}\right) \le 1$, even then the mode will be zero irrespective the value of β

4. Approximate Estimation of the parameters α and β

4.1. Maximum Likelihood Estimation (MLE) Method

Given a random sample $x_1, x_2, \ldots, x_n$, of size n from the PG distribution with p.m.f. (Equation-4) is

$$g(x;\alpha,\beta) = \left(\frac{\Gamma(x+\beta)}{x!\,\Gamma(\beta)}\right)\alpha^{\beta}(1+\alpha)^{-(x+\beta)}$$

$$= \left(\frac{\Gamma(x+\beta)}{x!\,\Gamma(\beta)}\right)\alpha^{\beta}(1+\alpha)^{-\beta}\quad(1+\alpha)^{-x} \qquad (17)$$

The likelihood function will be

$$P(x_i;\alpha,\beta) = \prod_{i=1}^{n} g(x_i;\alpha,\beta) \qquad (18)$$

The log likelihood becomes

$$L = \sum_{i=1}^{n}\log\left(\frac{\Gamma(x_i+\beta)}{x!\,\Gamma(\beta)}\right) + n\beta\log\alpha - n\beta\log(1+\alpha) - \log(1+\alpha)\sum_{i=1}^{n}x_i \qquad (19)$$

Here we get

$$\frac{\delta L}{\delta\alpha} = \frac{n\beta}{\alpha} - \frac{n\beta}{(1+\alpha)} - \frac{\sum_{i=1}^{n}x_i}{(1+\alpha)} \qquad (20)$$

and

$$\frac{\delta L}{\delta\beta} = \sum_{i=1}^{n}\left[\frac{\delta}{\delta\beta}\log\left(\frac{\Gamma(x_i+\beta)}{x!\,\Gamma(\beta)}\right)\right] + n\log\left(\frac{\alpha}{(1+\alpha)}\right) \qquad (21)$$

Solving (Equation-20) for α, we have

$\frac{\delta L}{\delta\alpha} = 0,$

Now we have

$$\hat{\alpha} = \frac{n\beta}{\sum_{i=1}^{n}x_i} = \frac{\beta}{\bar{x}} \qquad (22)$$

Putting this value of α in (Equation-21), we get

$$\sum_{i=1}^{n}\left[\frac{\delta}{\delta\beta}\log\left(\frac{\Gamma(x_i+\beta)}{x!\,\Gamma(\beta)}\right)\right] + n\log\left[\frac{n\beta}{\sum_{i=1}^{n}x_i + n\beta}\right] = 0 \qquad (23)$$

Here we are unable to get a direct solution for β. These equations (22) and (23) may be solved by a Numerical method. These estimators are also applicable for the distribution (Equation-4).

4.2. Method of Moments Estimation

The pmf of PGD are given as

$$g(x;\alpha,\beta) = \left(\frac{\Gamma(x+\beta)}{x!\,\Gamma(\beta)}\right)\alpha^{\beta}(1+\alpha)^{-(x+\beta)};$$

$$x = 0, 1, 2, \ldots \quad ;\ \alpha, \beta > 0 \qquad (24)$$

For the p.m.f. (Equation-6), we have

$$\mu_1' = \frac{\beta}{\alpha} \qquad (25)$$

$$\mu_2' = \frac{(\beta)(\alpha+\beta+1)}{\alpha^2} \qquad (26)$$

From (Equation-25), by replacing μ_1' by $\bar{x}$, we have

$\bar{x} = \frac{\beta}{\alpha}$

$$\hat{\alpha} = \frac{\beta}{\bar{x}}, \text{ which is same as MLE (Equation-22)} \qquad (27)$$

Now, we can write

$\mu_2' = \frac{\sum f_i x_i^2}{n} = \frac{(\beta)(\alpha+\beta+1)}{\alpha^2}$

Then

$$\hat{\beta} = \frac{\bar{x}^2}{\frac{\sum f_i x_i^2}{n} - \bar{x} - \bar{x}^2} \qquad (28)$$

For practical purposes we may take these estimators α and β for the population with pmf represented by (Equation-4).

5. Survival Function

Let F(k) be the cdf and f(k) is pmf of X. The survival function is given by

$S(k) = 1 - F(k) = \Pr\{X > k\} = \sum_{j=k+1}^{\infty} f(j)$, k= 1,2,…. (29)

with S(0) = 1. S may be defined over the whole non-negative real line by

$$S(t) = S(k) \quad \text{for } 0 \le k \le t < k+1, \quad k = 1,2,3,\ldots \qquad (30)$$

Where $t \in [0, \infty)$. Here S (t) is a right continuous function.

According to Lai (2013) our case is k =0,1,2…, that will be obtained by Y = X-1, and thus S(0-) = 1 and S(0) = pr (X = 0).

6. Classical Hazard Rate Function

Let hazard (failure) rate function h(k) defined as

$$h(k) = \Pr(X = k \mid X \ge k) = \frac{\Pr(X=k)}{\Pr(X \ge k)} = \frac{f(k)}{S(k-1)} \qquad (31)$$

provided

$\Pr(X \ge i) = 0$. It may be expressed as

$$h(x) = \frac{S(k-1) - S(k)}{S(k-1)} \qquad (32)$$

Equation (32) may be considered as the classical discrete hazard rate function. For convenience, we may simply refer it as the hazard rate function without the prefix 'classical'. (Lai, (2013))

7. Necessary and Sufficient Conditions: (Lai, (2013))

A sequence $\{h(k), k \ge 1\}$ is a discrete hazard rate if and only if

a. For all k < m, h(k) < 1 and h(m) = 1.The distribution is then defined over {1,2,…,m}, or
b. For all $k \in N^{+} = \{1, 2, \ldots\}$, $0 \le h(k) \le 1$ and $\sum_{i=1}^{\infty} h(i) = \infty$. The distribution is defined over $k \in N^{+}$ in this case (Shaked et al. (1995)).

It is easily verified that Hazard Rate obtained by (Eqation-32) for the distribution (Eqation-4).

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8. Applications

8.1. The following data set is due to Kemp and Kemp (1965) pertains to the distribution of mistakes in copying groups of random digits are given as :

No. of errors per groups :	0	1	2	3	4
Observed frequency :	35	11	8	4	2

Let us fit (Equation-4) this data using the method of moments. Here we get sample mean $\bar{x} = 0.783$, n = 60 and $\frac{\sum f_i x_i^2}{n} = 1.85$

Solving (Equation-27) and (Equation-28) we have, $\hat{\alpha} = 1.725007766$ and $\hat{\beta} = 1.35068108$

Using these estimators, we obtain the expected frequencies as shown in Table-1

Table 1: Chi-Square Goodness-of-fit test for the proposed model PGD (Equation-4)

X	o_i	e_i	$(o_i - e_i)^2$	$\frac{(o_i - e_i)^2}{e_i}$
0	35	32	9	0.2185
1	11	16	25	1.5625
2	8	7	1	0.142857142
3 4 5	4 2 =6 0	3 1= 5 1	1	0.2
	=60	=60		χ^2=2.186607142

Interpretation and conclusion

The calculated value of Chi-Square is equal to 2.186607142. The tabulated value of Chi-Square at 1 d.f. at 5 % level of significance is 3.841. From the results it is obvious that the calculated value of Chi-Square is less than the tabulated value of Chi-Square. So we can say that our proposed distribution is good fitted.

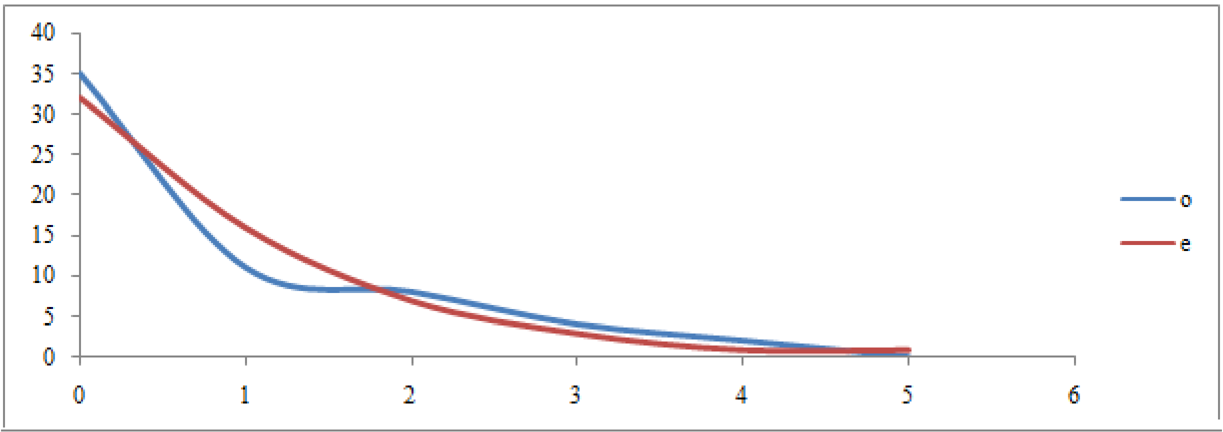

Figure 3: Observed and Fitted Frequency curves for table-1

8.2 When we choose β as integer equal to 1

The data set due to Kemp and Kemp (1965) as above may be used for the purpose of comparison

Since $\hat{\beta}$ 1.35061081, we can take $\hat{\beta}$ = 1.00 and $\hat{\alpha}$ = 1.277139208

Fitting the above data using $\hat{\alpha}$ = 1.277139208 and $\hat{\beta}$ = 1.00, we get the results as tabulated in Table-2 bellow

Table 2: Chi-Square Goodness-of-fit test for the proposed model PGD (Equation-6)

X	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	35	34	1	0.029411764
1	11	15	16	1.066666667
2	8	6	4	0.666666667
3 4 5	4 2 =6 0	3 1 =5 1	1	0.2
Total	60	60		χ^2 =1.962745098

Interpretation and conclusion

The calculated value of Chi-Square is equal to 1.962745098. The tabulated value of Chi-Square for 1 d.f. at 5 % level of significance is 3.841. Thus the calculated value of Chi-Square is less than the tabulated value of Chi-Square. So we can say that our proposed distribution is a good fit.

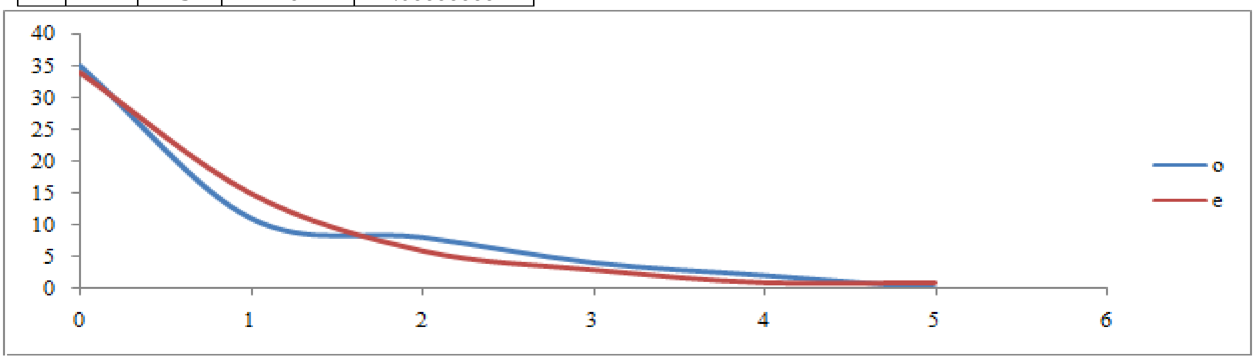

Figure 4: Observed and Fitted Frequency curves for table-2

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8.3 When we choose β as integer equal to 2

The data set due to Kemp and Kemp (1965) as above may be used for the purpose of comparison

Since $\hat{\beta}$ = 1.35061081, we can take $\hat{\beta}$ = 2.00 and $\hat{\alpha}$ = 2.554278416

Fitting the above data using $\hat{\alpha}$ = 2.554278416 and $\hat{\beta}$ = 2.00, we get the results as tabulated in Table-3 bellow.

Table 3: Chi-Square Goodness-of-fit test for the proposed model PGD (Eqation-6)

X	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	35	31	16	0.516129032
1	11	17	36	2.117647059
2	8	7	1	.142857142
3 4 5	4 2 = 6 0	3 1 = 5 1	1	0.2
Total	60	60		χ^2 =2.976633233

Interpretation and conclusion

The calculated value of Chi-Square is equal to 2.976633233 The tabulated value of Chi-Square for 1 d.f. at 5 % level of significance is 3.841. Thus the calculated value of Chi-Square is less than the tabulated value of Chi-Square. So we can say that our proposed distribution is a good fit.

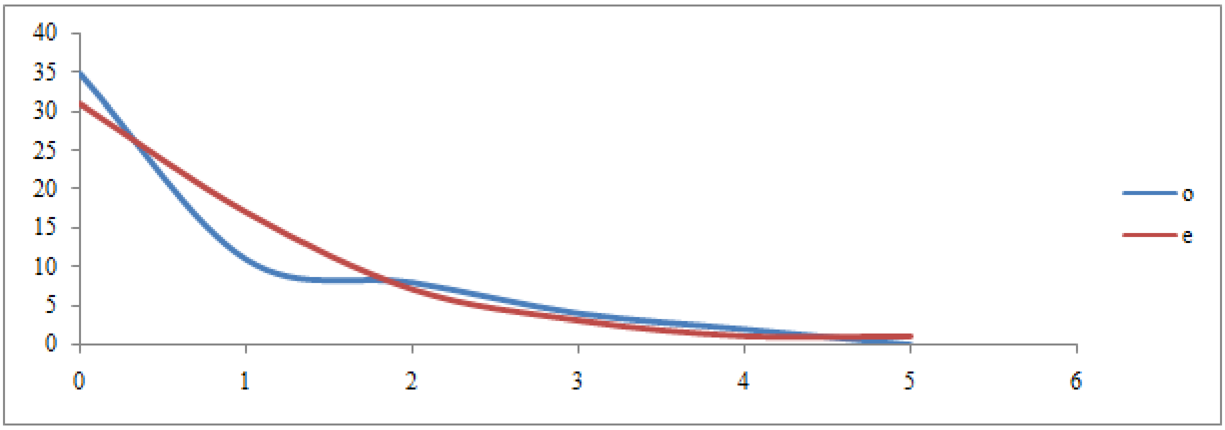

Figure 5: Observed and Fitted Frequency curves for table-3

8.4. This data set is Student's (1907) historic data on Haemocytometer counts of cells used by Borah (1984) for fitting the Gegenbauer distribution.

No. of yeast cells:	0	1	2	3	4	5	6
Observed frequency:	213	128	37	18	3	1	0

Let us fit (Eqation-4) using the method of moments. Here we get sample mean $\bar{x}$ = 0.6825, n = 400 and $\frac{\sum f_i x_i^2}{n}$ = 1.2275

Solving (Eqation-27) and (Eqation-28) we have, $\hat{\alpha}$ = 5.285064369 and $\hat{\beta}$ = 3.607056432. Using these estimators, we obtain the expected frequencies as shown in Table-4

Table 4: Chi-Square Goodness-of-fit test for the proposed model PGD (Equation-4)

X	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	213	214	1	0.004672897
1	128	123	25	0.203252032
2	37	45	64	1.422222222
3	18	13	25	1.923076923
4 5 6	3 1 = 4 0	4 1 = 5 0	1	0.2
Total =	400	400		χ^2 =3.75322236

Interpretation and Conclusion

The calculated value of Chi-Square is equal to 3.753222.36 The tabulated value of Chi-Square for 2 d.f. at 5 % level of significance is 5.99 Thus the calculated value of Chi-Square is less than the tabulated value of Chi-Square. So we can say that our proposed distribution is a good fit

Shankar, R.et al. (2012) has also fitted this data set by one parameter and two parameter Poisson-Lindley distribution and found that χ^2 = 14.3 for one parameter Poisson- Lindley distribution and χ^2 = 12.3 for two parameter Poisson Lindley distribution, which is not a good fit. It can be seen that our proposed distribution is better fit than the Shankar, R et al. (2012).

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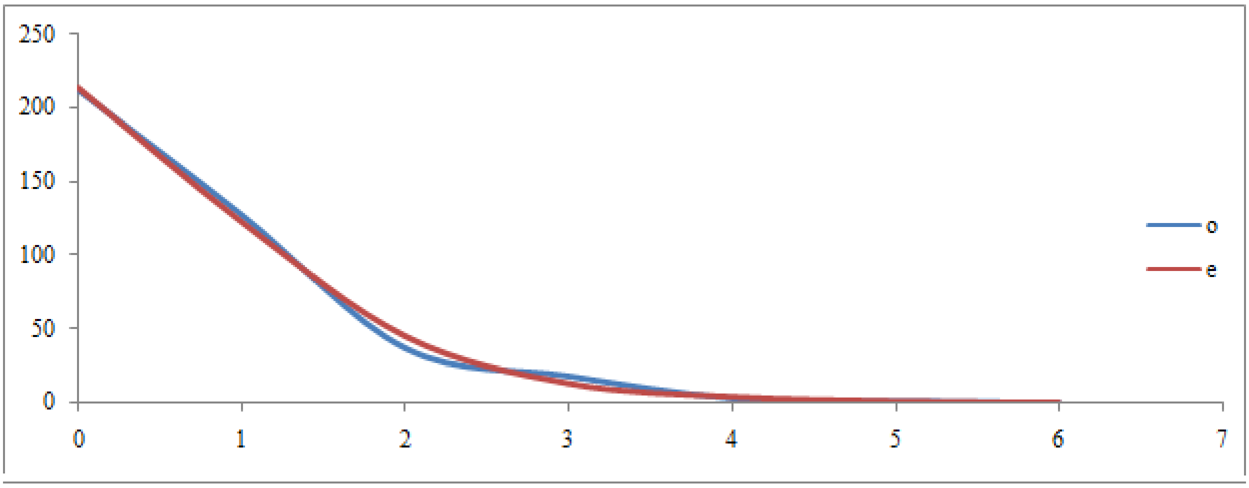

Figure 6: Observed and Fitted Frequency curves for table-4

8.5. When we choose β as integer equal to 3

This data set is Student's (1907) historic data on Haemocytometer counts of cells used by Borah (1984) for fitting the Gegenbauer distribution.

No. of yeast cells:	0	1	2	3	4	5	6
Observed frequency:	213	128	37	18	3	1	0

Let us fit (Eqation-6) using the method of moments. Here we get sample mean $\bar{x}$ = 0.6825, n = 400 and $\frac{\sum f_i x_i^2}{n}$ = 1.2275

Solving (Eqation-27) and (Eqation-28) we have, $\hat{\alpha}$ = 5.285064369 and $\hat{\beta}$ = 3.607056432

Let take the integral part of $\hat{\beta}$, i.e. $\hat{\beta}$ = 3

Using these estimators, we obtain the expected frequencies as shown in Table-5

Table 5: Chi-Square Goodness-of-fit test for the proposed model PGD (Equation-6)

X	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
0	213	216	9	0.041666667
1	128	120	64	0.533333333
2	37	45	64	1.422222222
3	18	14	16	1.142857143
4	3	4	1	0.2
5 6	1 = 4 0	1 = 5 0		
Total =	400	400		χ^2 =3.340079365

Interpretation and Conclusion

The calculated value of Chi-Square is equal to 3.340079365. The tabulated value of Chi-Square for 2 d.f. at 5 % level of significance is 5.99 Thus the calculated value of Chi-Square is less than the tabulated value of Chi-Square. So we can say that our proposed distribution is a good fit

Shankar, R.et al. (2012) has also fitted this data set by one parameter and two parameter Poisson-Lindley distribution and found that χ^2 = 14.3 for one parameter Poisson-Lindley distribution and χ^2 = 12.3 for two parameter Poisson Lindley distribution, which is not a good fit. It can be seen that our proposed distribution is better fit than the Shankar, R et al. (2012). We observe that the estimated value of β does not give a better fit than when β is taken as the nearest integer of its estimated value. There may be some cases when estimated value of α and β may give better results and the decision regarding the model to be used should be taken on the basis of goodness of fit criteria.

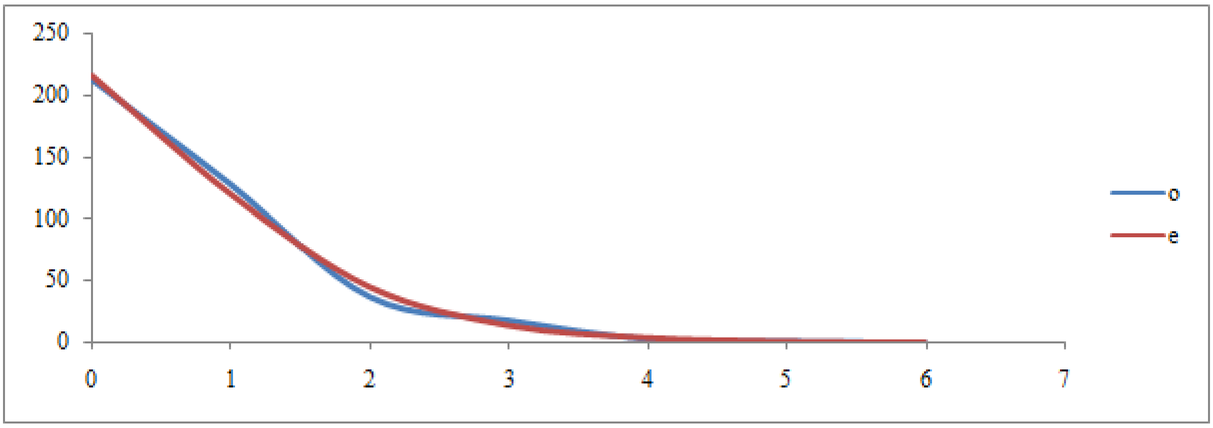

Figure 7: Observed and Fitted Frequency curves for table-5

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9. Conclusion

In view of the above discussions, we conclude that Poisson-Gamma Distribution (PGD) may be used as a discrete survival model.

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