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Fuzzy Subalgebras and Fuzzy p-ideals in TM-Algebras

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Abstract: ***In this study, we introduce the concepts of fuzzy subalgebras and fuzzy ideals in TM-algebras and investigate some of its properties.*** **Problem statement:** ***Let X be a TM-algebra, S be a sub algebra of X and I be a p-ideal of X. Let μ and ν be fuzzy sets in a TM-algebra X.*** **Approach: Define the upper level subset μ_t of μ and the cartesian product of μ and ν from X×X to [0,1] by minimum of μ (x) and ν (y) for all elements (x, y) in X×X. Result:** ***We proved any subalgebra of a TM-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X and μt is a p-ideal of X. Also we proved, the cartesian product of μ and ν is a fuzzy p-ideal of X×X.*** **Conclusion:** ***In this article, we have fuzzified the subalgebra and ideal of TM-algebras into fuzzy subalgrbra and fuzzy ideal of TM-algebras. It has been observed that the TM-algebra satisfy the various conditions stated in the BCC/ BCK algebras. These concepts can further be generalized.***

Keywords: TM-algebra, fuzzy sub algebra, fuzzy ideals, fuzzy p-ideal, homomorphism, Cartesian product, level subset, conditions stated.

1. Introduction

Isaki and Tanaka introduced two classes of abstract algebras BCI-algebras and BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebra. Hu and Li introduced a wide class of abstract algebra namely BCH- algebras. Zadeh (1965), introduced the notion of fuzzy sets in 1965. This concept has been applied to many mathematical branches. Xi applied this concept to BCK-algebra. Dudek and Jun (2001) fuzzified the ideals in BCC- algebras. Jun (2009) contributed a lot to develop the theory of fuzzy sets.

I introduced a new notion called TM-algebra, which is a generalization of Q/ BCK/ BCI/ BCH-algebra and investigated some properties. In this study, we introduce the concepts of fuzzy subalgebras and fuzzy p-ideals in TM-algebra and investigate some of their properties.

2. Materials Method

Certain fundamental definitions that will be used in the sequel are described.

Preliminaries:

Definition 1: A BCK-algebra is an algebra (X,*, 0) of type (2, 0) satisfying the following conditions:
- $(x* y) * (x * z) \leq z * y$
- $x* (x* y) \leq y$
- $x \leq x$,
- $x \leq y$ and $y \leq x$ imply $x = y$,
- $0 \leq x$ implies $x = 0$, where $x \leq y$ is defined by
- $x * y = 0$ for all x, y, z ∈ X.

Definition 2: Let I be a non- empty subset of a BCK- algebra X. Then I is called a BCK-ideal of X if:
- 0∈ I,
- $x * y \in I$ and $y \in I$ imply $x \in I$,for all x, y ∈ X

Definition 3: A TM-algebra (X, * ,0) is a non-empty set X with a constant "0" and a binary operation "* " satisfying the following axioms:
- $x * 0 = x$
- $(x* y) * (x * z) = z * y$, for any x, y, z ∈X

In X we can define a binary relation ≤ by $x \leq y$ if and only if $x * y = 0$.

Definition 4: Let S be a non-empty subset of a TM- algebra X. Then S is called a subalgebra of X if $x * y \in S$, for all x, y∈X .

Definition 5: Let (X,*, 0) be a TM-algebra. A non- empty subset I of X is called an ideal of X if it satisfies
- 0∈1
- $x * y \in I$ and $y \in I$ imply $x \in I$, for all x, y ∈ X .

Definition 6: An ideal A of a TM-algebra X is said to be closed if $0 * x \in A$ for all x ∈A .

Definition 7: Let (X,*,0) be a TM-algebra. A non- empty sub set I of X is called a p- ideal of X if it satisfies,
- 0 ∈ I
- $(x*z)*(y*z) \in I$ and $y \in I$ imply $x \in I$,y ∈ I , for all x, y, z ∈ X.

If we put z=0,then it follow that I is an ideal. Thus every p-ideal is an ideal.

3. Fuzzy Sub Algebras

Definition 8: Let X be a non-empty set. A mapping μ : x→ [0,1]. μ is called a fuzzy set in X. The complement of μ, denoted by $\bar{\mu}$ (x)= 1−μ(x), for all x ∈ X .

Definition 9: A fuzzy set μ in a TM-algebra X is called a fuzzy subalgebra of X if $\mu(x* y) \geq \min\{\mu(x), \mu(y)\}$, for all x, y ∈ X .

Definition 10: Let μ be a fuzzy set of a set X. For a fixed

$t \in [0,1]$, the set $\mu_t = \{ x \in X / \mu(x) \geq t\}$ is called an upper level of μ.

Fuzzy p-ideals in TM-algebras:

Definition 11: A fuzzy subset μ in a TM-algebra X is called a fuzzy ideal of X, if:

- (i) $\mu(0) \geq \mu(x)$
- (ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y, z \in X$

Definition 12: A fuzzy subset μ in a TM-algebra X is called a fuzzy p-ideal of X, if:

- $\mu(0) \geq \mu(x)$
- $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$, for all $x, y, z \in X$

4. Results

Lemma 13: If μ is a fuzzy subalgebra of a TM-algebra X, then $\mu(0) \geq \mu(x)$ for any $x \in X$.

Proof: Since $x * x = 0$ for any $x \in X$, then:
$\mu(0) = \mu(x*x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$.
This completes the proof.

Theorem 14: A fuzzy set μ of a TM-algebra X is a fuzzy subalgebra if and only if for every $t \in [0,1]$, μ_t is either empty or a subalgebra of X.

Proof: Assume that μ is a fuzzy subalgebra of X and $\mu_t \neq \varphi$. Then for any $x, y \in \mu_t$, we have: $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} \geq t$.
Therefore $x*y \in \mu_t$.
Hence μ_t is a subalgebra of X.
Conversely, μ_t is a subalgebra of X.
Let $x, y \in X$. Take $t = \min\{\mu(x), \mu(y)\}$.
Then by assumption μ_t is a sub algebra of X implies:
$x*y \in \mu_t$.
Therefore $\mu(x*y) \geq t = \min\{\mu(x), \mu(y)\}$.
Hence μ is a subalgebra of X.

Theorem 15: Any subalgebra of a TM-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.

Proof: Let μ be a subalgebra of a given TM-algebra X and let μ be a fuzzy set in X defined by:

$$\mu(x) = \begin{cases} t, if & x \in A \\ 0, if & x \in A \end{cases}$$

where, t (0,1) ∈ is fixed. It is clear that $\mu_t = A$.

Now we will prove that, such defined μ is a fuzzy subalgebra of X.
Let $x, y \in X$. If $x, y \in A$ then also $x*y \in A$.
Hence $\mu(x) = \mu(y) = \mu(x*y) = t$ and
$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$.
If $x, y \notin A$ then $\mu(x) = \mu(y) = 0$ and in the consequence
$\mu(x*y) \geq \min\{\mu(x), \mu(y)\} = 0$.
If at most one of x, y belongs to A,
then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0.
Therefore, $\min\{\mu(x), \mu(y)\} = 0$, so that:
$\mu(x*y) \geq 0$,
which completes the proof

Theorem 16: Two level subalgebras μs, μt (s < t) of a fuzzy subalgebra are equal if and only if there is no $x \in X$ such that $s \leq \mu(x) < t$.

Proof: Let $\mu_s = \mu_t$ for some $s < t$. If there exits $x \in X$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, which is a contradiction.
Conversely, assume that there is no $x \in X$, such that $s \leq \mu(x) < t$. If $x \in \mu_s$, then $\mu(x) \geq s$ and $\mu(x) \geq t$, since $\mu(x)$ does not lie between s and t.
Thus $\mu_t \in x$, which gives $\mu_s \subseteq \mu_t$. Also ts $\mu_t \subseteq \mu_s$.
Therefore $\mu_s = \mu_t$

Theorem 17: Every fuzzy p-ideal μ of a TM-algebra X is order reversing, that is if $x \leq y$ then:
$\mu(x) \geq \mu(y)$ for all $x, y \in X$.

Proof: Let $x, y \in X$ such that $x \leq y$
Therefore $x * y = 0$. Put $z = 0$,
Now, $\mu(x) = \mu(x*0)$
$\geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$
$= \min\{\mu((x*0) *(y*z)), \mu(y)\}$
$= \min\{\mu(x*y), \mu(y)\}$
$,\mu(x) = \mu(y)$.

Theorem 18: A fuzzy set μ in a TM-algebra X is a fuzzy p-ideal if and only if it is a fuzzy ideal of X.

Proof: Let μ be a fuzzy p-ideal of X. Then
(i) $\mu(0) \geq \mu(x)$ and
(ii) $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$
for all $x, y, z \in X$.
putting $z = 0$ in (ii) we have,
$\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$.
Hence μ is a fuzzy ideal of X.
Conversely, μ is a fuzzy ideal of X.
Then: $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y)\}$
$\mu(x) = \min\{\mu((x*z)*(y*z)), \mu(y)\}$
which proves the result.

Theorem 19: Let μ be a fuzzy set in a BCK-algebra X. Then μ is a fuzzy p-ideal if and only if μ is a fuzzy BCK-ideal.

Proof: Since every BCK-algebra is a TM-algebra, every fuzzy p-ideal is a fuzzy ideal of a TM-algebra and hence a fuzzy BCK-ideal.
Conversely, assume that μ be a BCK-ideal of X.
Then: $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$
$= \min\{\mu((x * z) * (y * z)), \mu(y)\}$.
Hence μ is a fuzzy p-ideal of X.

Theorem 20: Let μ be a fuzzy set in a TM-algebra X and let $t \in Im(\mu)$. Then μ is a fuzzy p-ideal of X if and only if the level subset: $\mu_t = \{ x \in X / \mu(x) \geq t \}$ is a p-ideal of X, which is called a level p-ideal of μ.

Proof: Assume that μ is a fuzzy p-ideal of X.
Clearly $0 \in \mu_t$
Let $((x * z) * (y * z)) \in \mu_t$ and $y \in \mu_t$.
Then $\mu((x* z) * (y * z)) \geq t$ and $\mu(y) \geq t$.
Now $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$
$\geq \{t, t\} = t$
Hence μ_t is p-ideal of X.
Conversely, let μ_t is p-ideal of X for any $t \in [0,1]$.
Suppose assume that there exist some $x_0 \in X$ such that
$\mu(0) < \mu(x_0)$;
Take $S = \frac{1}{2}[\mu(0) + \mu(x_0)]$
$\Rightarrow s < \mu(x_0)$ and $0 \leq \mu(0)\, s \leq 1$

$x_0 \in \mu_t$ and0 $\mu \notin \mu_t$ a contradiction,
since μ_s is a p-ideal of X.
Therefore, $\mu(0) \geq \mu(x)$ for all $x \in X$.
If possible, assume that $x_0, y_0\ z_0 \in X$
such that $\mu(x_0) \geq \min\{\mu((x_0 * z_0) * (y_0 * z_0)), \mu(y_0)\}$:
Take
$s = \frac{1}{2}[\mu(x_0) + \mu((x_0*z_0)*(y_0*z_0)), \mu(y_0)]$
$\Rightarrow s > \mu(x_0)$ and: $s < \min\{\mu((x_0*z_0)*(y_0*z_0)), \mu(y_0)\}$
$\Rightarrow s > \mu(x_0)$, $s < \mu((x_0*z_0)*(y_0*z_0))$ and $s < \mu(y_0)$
$x_0 \notin \mu_s \Rightarrow$, a contradiction, since μ_s is a p-ideal of X.
Therefore, $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y)\}$
for any $x, y, z \in X$.

5. Cartesian product of fuzzy p-ideals of TM-algebras:

Definition 21: Let μ and ν be the fuzzy sets in a set X. The Cartesian product
$\mu \times \nu: X \times X \to [0,1]$ is defined by:
$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ for all $x, y \in X$

Theorem 22: If μ and ν are fuzzy p-ideals in a TM- algebra X, then $\mu \times \nu$ is a fuzzy p-ideal in X×X.

Proof: For any $(x, y) \in X\times X$, we have:

$(\mu \times \nu)(0, 0) = \min\{\mu(0), \nu(0)\}$
$\geq \min\{\mu(x), \nu(y)\}$
$= (\mu \times \nu)(x, y)$.

Let $(x_1, x_2), (y_1, y_2)$ and $(z_1, z_2) \in X\times X$.

$(\mu \times \nu)(x_1, x_2) = (\mu \times \nu)(x_1 * x_2)$
$= \min\{\mu(x_1), \nu(x_2)\}$

$\geq \min\{\min\{\mu((x_1*z_1)*(y_1*z_1)), \mu(y_1)\}, \min\{\nu((x_2*z_2)*(y_2*z_2)), \nu(y_2)\}\}$
$= \min\{\min\{\mu((x_1*z_1)*(y_1*z_1)), \nu((x_2*z_2)*(y_2*z_2)), \min\{\mu(y_1), \nu(y_2)\}\}$
$= \min\{(\mu\times\nu)((x_1*z_1)*(y_1*z_1)), \nu((x_2*z_2)*(y_2*z_2)), (\mu\times\nu)(y_1, y_2)\}$
$= \min\{(\mu\times\nu)((x_1*z_1)*(y_1*z_1)), \nu((x_2*z_2)*(y_2*z_2)), (\mu\times\nu)(y_1, y_2)\}$

Theorem 23: Let μ and ν be fuzzy sets in a TM-algebra X such that $\mu \times \nu$ is a Hence $\mu \times \nu$ is a fuzzy p-ideal of a TM-algebra in X×X. fuzzy p-ideal of X×X. Then:

- (i) Either $\mu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$ for all $x \in X$
- (ii) If $\mu(0) \geq \mu(x)$ for all $x \in X$, then either $\nu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$
- (iii) If $\nu(0) \geq \nu(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq \nu(x)$
- (iv) Either μ or ν is a fuzzy p-ideal of X.

Proof: $\mu \times \nu$ is a fuzzy p-ideal of X×X. Therefore
$(\mu\times\nu)(0, 0) \geq (\mu \times \nu)(x, y)$ for all $(x, y) \in X \times X$
And
$(\mu \times \nu)(x_1, x_2) \geq \min\{(\mu \times \nu)((x_1, x_2) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2)), (\mu\times\nu)(y_1, y_2)\}$ for all $(x_1, x_2), (y_1, y_2)$ and $(z_1, z_2) \in X\times X$.

Suppose that $\mu(0) < \mu(x)$ and $\nu(0) < \nu(y)$ for some $x, y \in X$.

Then: $(\mu \times \nu)(x,y) = \min\{\mu(x), \nu(y)\}$
$> \min\{\mu(0), \nu(0)\}$
$= (\mu\times\nu)(0,0)$.

a contradiction.

Therefore either $\mu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$ for all $x \in X$.

Assume that there exist $x, y \in X$ such that:
$\nu(0) < \mu(x)$ and $\nu(0) < \nu(y)$.

Then: $(\mu\times\nu)(0,0) = \min\{\mu(0), \nu(0)\}$
$= \nu(0)$ and hence
$(\mu\times\nu)(x, y) = \min\{\mu(x), \nu(y)\} > \nu(0)$
$= (\mu\times\nu)(0,0)$,

a contradiction.

Hence if $\mu(0) \geq \mu(x)$ for all $x \in X$,
then either: $\nu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$
Similarly we can prove that if $\nu(0) \geq \nu(x)$ for all $x \in X$, then either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq \nu(x)$. First we prove that ν is a fuzzy p-ideal of X. Since ,by (i),either $\mu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$ for all $x \in X$. Assume that $\nu(0) \geq \nu(x)$ for all $x \in X$. It follows from (iii) that either $\mu(0) \geq \mu(x)$ or $\mu(0) \geq \nu(x)$.If$\mu(0) \geq \nu(x)$
for any $x \in X$, then:

$\nu(x) = \min\{\mu(0), \nu(x)\}$
$= (\mu\times\nu)(0, x)$.
$\nu(x) = \min\{\mu(0), \nu(x)\}$
$= (\mu\times\nu)(0, x)$
$\geq \min\{(\mu\times\nu)((0, x)*(0, z))*((0*y)*(0, z)), (\mu\times\nu)(0, y)\}$
$= \min\{(\mu\times\nu)(((0*0),(x*z))*((0*0),(y*z))), (\mu\times\nu)(0, y)\}$
$= \min\{(\mu\times\nu)(((0*0)*(0*0)),((x*z)*(y*z))), (\mu\times\nu)(0, y)\}$
$= \min\{(\mu\times\nu)(0, ((x*z)*(y*z))), (\mu\times\nu)(0,y)\}$
$V(x) = \min\{\nu((x*z)*(y*z)), \nu(y)\}$.

Hence ν is a fuzzy p-ideal of X.
Now we will prove that μis a fuzzy p-ideal of X.
Let $\mu(0) \geq \mu(x)$. By (ii) either $\nu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(x)$.
Assume that $\nu(0) \geq \mu(x)$,
Then:
$\mu(x) = \min\{\mu(x), \nu(0)\} = (\mu\times\nu)(x,0)$.
$\mu(x) = \min\{\mu(x), \nu(0)\}$
$= (\mu\times\nu)(x, 0)$
$\geq \min\{(\mu\times\nu)(((x,0)*(z,0))*((y*0)*(z,0))), (\mu\times\nu)(y, 0)\}$
$= \min\{(\mu\times\nu)(((x*z),(0*0))*((y*z),(0*0))), (\mu\times\nu)(y, 0)\}$
$= \min\{(\mu\times\nu)(((x*z)*(y*z)),((0*0)*(0*0))), (\mu\times\nu)(y, 0)\}$
$= \min\{\mu((x*z)*(y*z)), \mu(y)\}$

Hence μ is a fuzzy p-ideal of X.

6. Homomorphism of TM-algebras:

Definition 24: Let X and Y be TM-algebras. A mapping $f: X \to Y$ is said to be a homomorphism if it satisfies: $f(x*y) = f(x)*f(y)$, for all $x, y \in X$.

Definition 25: Let f: X x X → be an endomorphism and μ a fuzzy set in X. We define a new fuzzy set in X by μ_f in X by $\mu_f(x) = \mu(f(x))$, for all x in X.

Theorem 26: Let f be an endomorphism of a TM- algebra X. If μ is a fuzzy p-ideal of X, then so is μ_f.

Proof: $\mu_f(x) = \mu(f(x)) \leq \mu(0)$
Let $x, y, z \in X$.
Then: $\mu_f(x) = \mu(f(x))$
$\geq \min\{\mu((f(x)*f(z))*(f(y)*f(z))), \mu(f(y))\}$
$= \min\{\mu((f(x*z))*f(y*z)), \mu(f(y))\}$
$= \min\{\mu(f((x*z)*(y*z))), \mu(f(y))\}$

$= \min \{\mu_f ((x*z)*(y* z)) , \mu_f (y)) \}$.

Hence μ_f is a fuzzy p-ideal of X.

7. Discussion

With minimum conditions in TM-algebra it satisfy these results. In other algebra Like BCK/BCI/BCH/BCC the number of conditions are more.

8. Conclusion

In this article, we have fuzzified the subalgebra and ideal of TM-algebras into fuzzy subalgebra and fuzzy ideal of TM-algebras. It has been observed that the TM-algebra satisfy the various conditions stated in the BCC/ BCK algebras and can be considered as the generalization of all these algebras. These concepts can further be generalized.

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