International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2013): 4.438

The Dual Activation of the Geometric Inverse Burr Distribution: Comparison Between Maximum and Minimum

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Abstract: ***In this paper, we introduced a double activation approach for the geometric inverse burr distribution. Statistical measures and their properties are derived. Explicit expression for their density, rth moment and entropy are obtained. The method of maximum likelihood estimator is used to obtain the estimate values of the parameters and the information matrixs are provided. Comparison between minimum and maximum for the simulation studies are performed for different parameter values and sample sizes to assess their finite sample behavior of the MLEs.***

Keywords: probability density function(pdf), probability mass function(pmf), Inverse Burr(IB), geometric inverse burr(GIB), hazard function, cumulative distribution function(cdf), moments, maximum likelihood estimation(MLEs) and quartile.

1. Introduction

The inverse Burr distribution is been used in various fields of sciences. In the actuarial literature it is known as the Burr III distribution (see, e.g., Klugman et al., 1998) and as the kappa distribution in the meteorological literature (Mielke, 1973; Mielke and Johnson, 1973). It has also been used in finance, environmental studies, survival analysis and reliability theory (see Sherrick et al., 1996; Lindsay et al., 1996; Al-Dayian, 1999; Shao, 2000; Hose, 2005; Mokhlis, 2005; Gove et al., 2008). Further, Shao et al. (2008) proposed an extended inverse Burr distribution in low-flow frequency analysis where its lower tail is of main interest. A bivariate extension of the inverse Burr distribution had been given by Rodriguez (1980). The cumulative distribution function(cdf) of the Inverse Burr is given by

$$F_{\alpha,\beta} = \left(\frac{w^{\alpha}}{1+w^{\alpha}}\right)^{\beta}.$$

$where\ \alpha > 0, \beta > 0\ are\ shape\ parameters.$

While the pdf of the inverse burr is given by

$$f_{\alpha,\beta}(w) = \alpha\beta w^{\alpha-1}\left(\frac{w^{\alpha}}{1+w^{\alpha}}\right)^{\beta-1}(1+w^{\alpha})^{-2}$$

Different authors have introduced a new distributions in the literature to model lifetime data by combining geometric and other well known distributions. Adamidis and Loukas (1998) introduced the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus (2007) introduced the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Marshall and Olkin (1997) presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. (2005) proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the EEG distribution can be monotone decreasing, increasing or constant. Wagnar et al. (2008) proposed the weibull geometric distribution. Recently, Abdullahi et al. (2015) introduced the geometric inverse burr distribution by combining geometric and inverse burr distribution to form geometric inverse burr distribution(GIB), but in their paper they approached the distribution base on single approach that is the minimum. In this paper, we introduced a new approach to the distribution of Abdullahi et al.(2015) based on double approach that is the minimum and maximum . A comparison for the simulation studies was performed between the Maximum and the Minimum that is between GIB 1 and GIB 2 and GIB 2 gave better finite sample behavior of the MLEs.

The rest of the paper is organized as follows. Section 2, provides the new distributions. Statistical properties of this class of distribution are given in section 3. Statistical inferences and entropy are given in Sections 4 and 5 respectively. Section 6 gives the simulation studies for different parameter values and sample sizes. Conclusions of the paper are given in Section 7.

2. The dual activation for the GIB distribution

In this section, we introduce the new class of distribution call the geometric inverse burr distribution(GIB) base on dual activation i.e GIB 1 and GIB 2. The pdf of the GIB 1 and GIB 2 are a decreasing and unimodal, while the hazard rate functions are decreasing, increasing and a bathtube shape depending on the parameter values. The Figures below have clearly shown the shape of the pdf and the hazard rate function for the distributions.

• Let z be a geometric random variable with pmf $P(z,p) = (1-p)p^{z-1}$ for $z \in \mathbb{N}$ and $p \in (0,1)$. Define $X_1 = max(w_1,...,w_z)$. The marginal pdf of X

is

$$f(x;p,\alpha,\beta)=\frac{\alpha\beta(1-p)x^{-(\alpha+1)}\left(1+x^{-\alpha}\right)^{-(\beta+1)}}{\left[1-p(1+x^{-\alpha})^{-\beta}\right]^{2}}\quad x>0$$

(1)

which defines the probabilty density function of the GIB1. The cdf for the GIB1 becomes

$$F_{\alpha,\beta}(x;p,\alpha,\beta)=(1-p)\left(\frac{x^{\alpha}}{1+x^{\alpha}}\right)^{\beta}\left[1-p(1+x^{-\alpha})^{-\beta}\right]^{-1}\quad x>0$$

(2)

The hazard rate function of the GIB1 is

$$h(x,p,\alpha,\beta)=\frac{\alpha\beta(1-p)x^{-(\alpha+1)}(1+x^{-\alpha})^{-(\beta+1)}\left[1-p(1+x^{-\alpha})^{-\beta}\right]^{-1}}{\left[1-p(1+x^{-\alpha})^{-\beta}\right]-(1+x^{-\alpha})^{-\beta}}\quad x>0$$

(3)

The survival function of the GIB1 is

$$s(x,p,\alpha,\beta)=1-(1-p)\left(\frac{x^{\alpha}}{1+x^{\alpha}}\right)^{\beta}\left[1-p(1+x^{-\alpha})^{-\beta}\right]^{-1}\quad x>0$$

(4)

- Let z be a geometric random variable with pmf $P(z,p)=(1-p)p^{z-1}$ for $z\in\mathbb{N}$ and $p\in(0,1)$.

Define $X_2=min(w_1,\ldots,w_z)$. The marginal pdf of X is

$$f(x;p,\alpha,\beta)=\frac{\alpha\beta(1-p)x^{-(\alpha+1)}\left(1+x^{-\alpha}\right)^{-(\beta+1)}}{\left[1-p(1-(1+x^{-\alpha})^{-\beta})\right]^{2}}\quad x>0$$

(5)

which defines the pdf of the GIB 2. The cdf for the GIB 2 becomes

$$F_{\alpha,\beta}(x;p,\alpha,\beta)=\left(\frac{x^{\alpha}}{1+x^{\alpha}}\right)^{\beta}\left[1-p(1-(1+x^{-\alpha})^{-\beta})\right]^{-1}\quad x>0$$

(6)

The hazard rate function of the GIB 2 is

$$h(x,p,\alpha,\beta)=\frac{\alpha\beta(1-p)x^{-(\alpha+1)}(1+x^{-\alpha})^{-(\beta+1)}\left[1-p(1-(1+x^{-\alpha})^{-\beta})\right]^{-1}}{\left[1-p(1-(1+x^{-\alpha})^{-\beta})\right]-(1-p)(1+x^{-\alpha})^{-\beta}}\quad x>0 \quad (7)$$

The survival function of the GIB 2 is

$$s(x,p,\alpha,\beta)=1-(1-p)\left(\frac{x^{\alpha}}{1+x^{\alpha}}\right)^{\beta}\left[1-p(1-(1+x^{-\alpha})^{-\beta}\right)$$

(8)

- Graph of the pdf and the hazard rate function for the GIB1

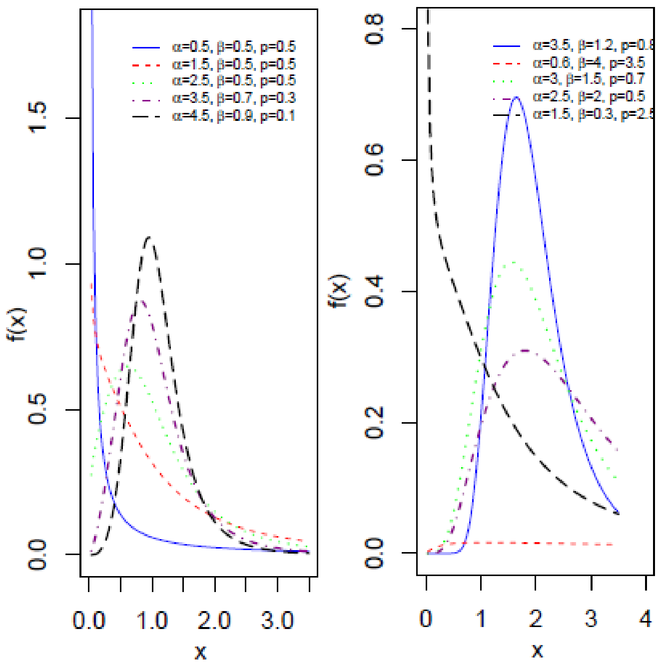

Figure 1: pdf for GIB1 for different parameter values

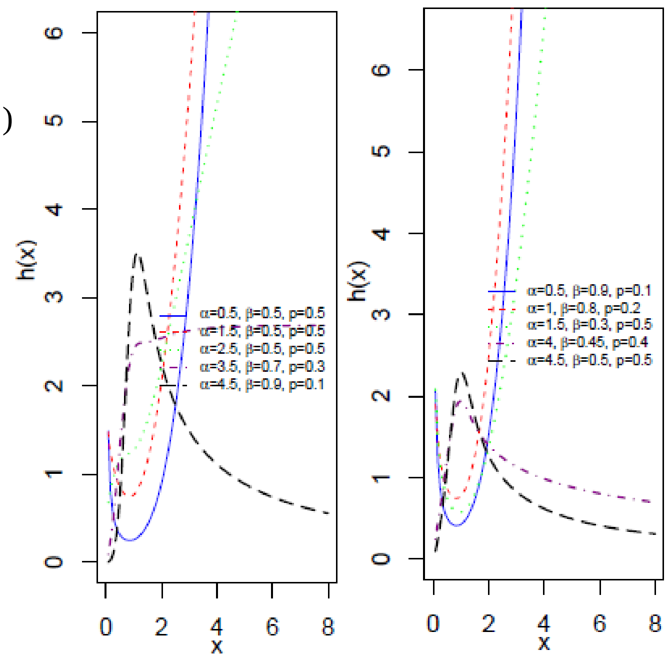

Figure 2: hazard rate function for GIB1 for different parameter values

- Graph of the pdf and the hazard rate function for the GIB 2

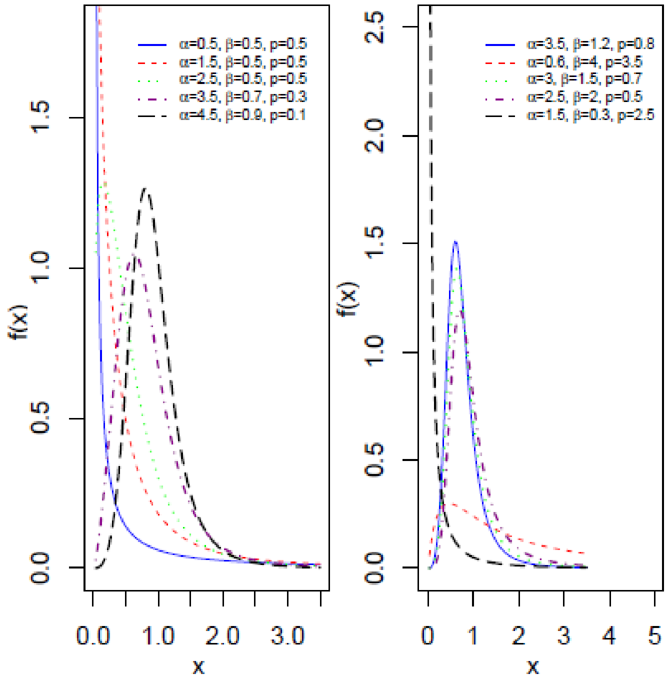

Figure 3: pdf for GIB2 for different parameter value

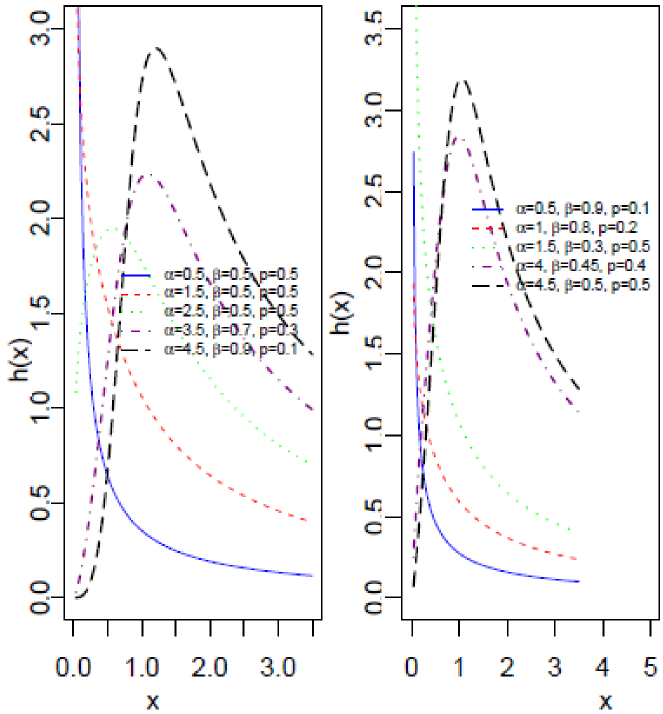

Figure 4: hazard rate function for GIB2 for different parameter values

The following propositions holds for both the GIB1 and the GIB2

Proposition 2.1 *The pdf for the GIB1 and GIB2 is decreasing for* $0<\alpha<1$ *and unimodal for* $\alpha>1$.

Proposition 2.2 *The limiting behaviour of the hazard rate function for both the GIB1 and GIB2 distributions is as follows*

if $0<\beta<1$, then

$$\lim_{x\to 0} h(x)=\begin{cases}\infty & \text{if } 0<\alpha\le 1\\ 0 & \text{if } \alpha>0\end{cases} \quad (9)$$

and $\lim_{x\to\infty} h(x)=0$

for $\beta=1$,

$$\lim_{x\to 0} h(x)=\begin{cases}\infty & \text{if } 0<\alpha<1\\ \dfrac{1}{1-p} & \alpha=1\\ 0 & \text{if } \alpha>1\end{cases} \quad (10)$$

and $\lim_{x\to\infty} h(x)=0$

for $\beta>1$, $\lim_{x\to 0} h(x)=0$, for each value $\alpha>0$ $\lim_{x\to\infty} h(x)=\infty$

Proposition 2.3 *The hazard function is decreasing for* $0<\alpha\le 1$ *and for* $\alpha>1$ *it can take different forms.*

Now, for $|z|<1$ and $\rho>0$, the power series expansion is given by

$$(1-z)^{-\rho}=\sum_{j=0}^{\infty}\frac{\Gamma(p+j)z^j}{\Gamma(p)\,j!} \quad (11)$$

We use (11) for the derivation of the properties of the geometric inverse burr(GIB1 and GIB2). Using (11) in equation (4.1) we obtain the following

- for GIB1

$$f(x;p,\alpha,\beta)=(1-p)\sum_{j=0}^{\infty}\alpha\beta(j+1)x^{-(\alpha+1)}\left(1+x^{-\alpha}\right)^{-\beta(j+1)-1}p^j \quad (12)$$

where $\alpha(j+1)x^{-(\alpha+1)}\left(1+x^{-\alpha}\right)^{-\beta(j+1)-1} = f_{\alpha,B(j+1)}(x)$

now, letting $v_i=(1-p)p^j$, equation (12) becomes

$$f(x;p,\alpha,B)=\sum_{i=0}^{\infty}v_i\, f_{\alpha,B(j+1)}(x) \quad (13)$$

where $f_{\alpha,B(i+1)}(x)$ is the pdf of the inverse burr.

- for GIB2

Using (11) in equation (4.1) and thereafter applying binomial expansion, we obtain the following

$$f(x;p,\alpha,\beta)=(1-p)\sum_{i=0}^{\infty}(-1)^i(i+1)^{-1}\sum_{j=i}^{\infty}(j+1)p^j\binom{j}{i}f_{\alpha,B(i+1)}(x) \quad (14)$$

by letting

$$m_i=(1-p)(-1)^i(i+1)^{-1}\sum_{j=i}^{\infty}(j+1)p^j\binom{j}{i},$$

equation (14) becomes

$$f(x;p,\alpha,B)=\sum_{i=0}^{\infty}m_i f_{\alpha,B(i+1)}(x) \quad (15)$$

where $f_{\alpha,B(i+1)}(x)$ is the pdf of the inverse burr.

3. Statistical Properties

In this section, we discuss some of the statistical properties for GIB. Among which includes the following

3.1 Moments

Theorem 3.1 *If* $Z:IB(\alpha,\beta)$, *the* $(r,n)^{th}$ *probability weighted moment(pwm) of* Z *becomes*

$$m(r,n)=aB(a(n+1)+\frac{r}{\alpha},1-\frac{r}{\alpha}).$$

where $a=\beta$

Proof. Similar to that of GIB 2 given in appendix Ai and so is omitted.

• rth moment,mean and variance for GIB1

Theorem 3.2 *If* $X_1:GIB1(p,\alpha,\beta)$, *the* r^{th} *moment* (pwm) *of* X_1 *is given by*

$$E(X^r)=\beta\sum_{j=0}^{\infty}(j+1)n_jB(\beta(j+1)+\frac{r}{\alpha},1-\frac{r}{\alpha}).$$

Proof. Similar to that of GIB 2 given in appendix Aii and so is omitted.

3.2 Mean and Variance

$$E(X)=\beta\sum_{j=0}^{\infty}(j+1)n_iB(\beta(j+1)+\frac{1}{\alpha},1-\frac{1}{\alpha})$$

$$E(X^2)=\beta\sum_{j=0}^{\infty}(j+1)n_iB(\beta(j+1)+\frac{2}{\alpha},1-\frac{2}{\alpha})$$

$$Var(X)=\beta\sum_{j=0}^{\infty}(j+1)n_iB(\beta(j+1)+\frac{2}{\alpha},1-\frac{2}{\alpha})-\left\{\beta\sum_{j=0}^{\infty}(j+1)n_iB(\beta(j+1)+\frac{1}{\alpha},1-\frac{1}{\alpha})\right\}^2$$

• rth moment, mean and variance for GIB 2

Theorem 3.3 *If* $X_2:GIB2(p,\alpha,\beta)$, *the* r^{th} *moment* (pwm) *of* X_1 *is given by*

$$E(X^r)=a\sum_{i=0}^{\infty}m_i(i+1)B(a(i+1)+\frac{r}{\alpha},1-\frac{r}{\alpha}).$$

where $a=\beta$

Proof. See Appendix Aii

3.3 Mean and Variance

$$E(X)=a\sum_{i=0}^{\infty}m_i(i+1)B(a(i+1)+\frac{1}{\alpha},1-\frac{1}{\alpha})$$

$$E(X^2)=a\sum_{i=0}^{\infty}m_i(i+1)B(a(i+1)+\frac{2}{\alpha},1-\frac{2}{\alpha})$$

$$Var(X)=a\sum_{i=0}^{\infty}m_i(i+1)B(a(i+1)+\frac{2}{\alpha},1-\frac{2}{\alpha})-\left\{a\sum_{i=0}^{\infty}m_i(i+1)B(a(i+1)+\frac{1}{\alpha},1-\frac{1}{\alpha})\right\}^2$$

3.4 Quatile and Median

• by inverting the cdf of the GIB1 we obtained the quantile function $(for\ 0<q<1)$ as

$$x_q=\left\{\left(\frac{1-p(1-q)}{q}\right)^{\frac{1}{\beta}}-1\right\}^{-(\frac{1}{\alpha})} \quad (16)$$

(16) is used for the simulation of the GIB1. Therefore, we can have the median as

$$x_{0.5}=\left\{((2-p))^{\frac{1}{\beta}}-1\right\}^{-(\frac{1}{\alpha})} \quad (17)$$

• by inverting the cdf of the GIB 2 we obtained the quantile function $(for\ 0<q<1)$ as

$$x_q=\left\{\left(\frac{1-pq}{q(1-p)}\right)^{\frac{1}{\beta}}-1\right\}^{-(\frac{1}{\alpha})} \quad (18)$$

(18) is used for the simulation of the GIB 2. Therefore, we can have the median as

$$x_{0.5} = \left\{ \left(\frac{2-p}{(1-p)} \right)^{\frac{1}{\beta}} - 1 \right\}^{-(\frac{1}{\alpha})} \quad (19)$$

The skewness and kurtosis for both the GIB1 and GIB2 can be obtained from the following equations respectively.

$$\gamma_3 = \frac{\mu^{(3)} - 3\mu\mu^{(2)} + 2\mu^3}{\left(\mu^{(2)} - \mu^2\right)^{\frac{3}{2}}} \quad (20)$$

$$\gamma_4 = \frac{\mu^{(4)} - 4\mu\mu^{(3)} + 6\mu^2\mu^{(2)} - 3\mu^4}{\left(\mu^{(2)} - \mu^2\right)^2} \quad (21)$$

4. Statistical Inference

In this section, we discuss the estimation problem about the unknown parameters of the proposed model. For the estimation problem, we discuss the most popular method of estimation used in statistical science namely, the method of maximum likelihood estimators(MLEs). This is because the MLEs possesses under fairly regular condition of some optimal properties.

4.1 Estimation for GIB1

Let $X_1,..., X_n$ be a random sample with observed values $x_1,..., x_n$ from the class with parameters p, α, β. Let $\Theta = (p, \alpha, \beta)^T$ be the parameter vector.The log likelihood function is given by

$$l(\theta) = \log \alpha\beta(1-p) - (\alpha+1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log(1+x_i^{-\alpha}) - 2\sum_{i=1}^{n}\log\left(1-p(1+x^{-\alpha})\right) \quad (22)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equation obtained by differentiating $l(x; p, \alpha, \beta)$ above. The components of the score vector $U = (\frac{\partial l}{\partial p}, \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta})^T$ are given by

$$\frac{\partial l}{\partial p} = -\frac{n}{1-p} - 2\sum_{i=1}^{n}\frac{\left(1+x_i^{-\alpha}\right)^{-(\beta)}}{\left(1-pt_i^{-\beta}\right)} \quad (23)$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log x_i + \alpha(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i} + 2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)} \quad (24)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log t_i + 2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)} \quad (25)$$

where $t_i = (1+x_i^{-\alpha})$.

The 3×3 observed information matrix is given by

$$\mathbf{J} = \begin{pmatrix} J_{PP} & J_{p\alpha} & J_{p\beta} \\ J_{\alpha\beta} & J_{\alpha\alpha} & J_{\alpha P} \\ J_{\beta p} & J_{\beta\alpha} & J_{\beta\beta} \end{pmatrix}$$

where the expressions for the elements of J are given in Appendix B

The following theorem is applied for both GIB1 and GIB2

Theorem 4.1 *The maximum likelihood estimators* $(\hat{p}, \hat{\alpha}, \hat{\beta})$ *are consistent estimators, and* $\sqrt{n}\left(\hat{p}-p, \hat{\alpha}-\alpha, \hat{\beta}-\beta\right)^T$ *is asymptotically normal with mean vector* $\mathbf{0}$ *and the variance covariance matrix* $\mathbf{I}^{-1}$, *where* $\mathbf{I} = -\frac{1}{n}E\left(\frac{\partial l_1^2(\Theta)}{\partial\Theta\partial\Theta^T}\right)$.

(4.1) is applied for both GIB1 and GIB2 respectively.

4.2 Estimation for GIB2

Let $X_1,..., X_n$ be a random sample with observed values $x_1,..., x_n$ from the class with parameters p, α, β. Let $\Theta = (p, \alpha, \beta)^T$ be the parameter vector.The log likelihood function is given by

$$l(\theta) = \log \alpha\beta(1-p) - (\alpha+1)\sum_{i=1}^{n}\log x_i - (\beta+1)\sum_{i=1}^{n}\log(1+x_i^{-\alpha}) - 2\sum_{i=1}^{n}\log\left(1-p[1-(1+x^{-\alpha})]\right) \quad (26)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equation obtained by differentiating $l(x; p, \alpha, \beta)$ above. The components of the score vector $U = (\frac{\partial l}{\partial p}, \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta})^T$ are given by

$$\frac{\partial l}{\partial p} = -\frac{n}{1-p} - 2\sum_{i=1}^{n}\frac{-1+\alpha\beta x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)} \quad (27)$$

$$\frac{\partial l}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\log x_i+\alpha(\beta+1)\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i^{-(\beta+1)}}-2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}-t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)} \quad (28)$$

$$\frac{\partial l}{\partial \beta}=-\frac{n}{\beta}-\sum_{i=1}^{n}\log t_i+2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)}$$
(29)

where $t_i=(1+x_i^{-\alpha})$

The 3×3 observed information matrix is given by

$$\mathbf{J}=\begin{pmatrix} J_{PP} & J_{p\alpha} & J_{p\beta}\\ J_{\alpha\beta} & J_{\alpha\alpha} & J_{\alpha P}\\ J_{\beta p} & J_{\beta\alpha} & J_{\beta\beta}\end{pmatrix}$$

where the expressions for the elements of J are given in Appendix B

5. Entropy

Statistical entropy is a probabilistic measure of uncertainty about the outcome of a random experiment, and is a measure of a reduction in that uncertainty. Numerous entropy and information indices, among them the Renyi entropy, have been developed and used in various disciplines and contexts. Information theoretic principles and methods have become integral parts of probability and statistics and have been applied in various branches of statistics and related fields. Entropy has been used in various situations in science and Engineering. The entropy of a random variable X is a measure of variation of the uncertainty. Renyi entropy is defined by

$$I_R(r)=\frac{1}{1-r}\log\left[\int_R f^r(x)dx\right] \quad (30)$$

where $r>0$ and $r\neq 1$

• for GIB1 we have,

$$\int_R f^r(x)dx=\alpha^{r-1}(\beta(1-p))^r\sum_{j=0}^{\infty}(j+1)p^j\int_0^{\infty}f_{\alpha,B(i+1)}(x)$$

where

$$\int_0^{\infty}f_{\alpha,B(i+1)}(x)=\frac{1}{\alpha}B\left(a(i+r)+r-1-\frac{(\alpha+1)(r-1)}{\alpha},1+\frac{(\alpha+1)(r+1)}{\alpha}\right)$$

.

Consequentlty,

$$I_R(r)=\frac{1}{1-r}\log[\alpha^{r-1}(\beta(1-p))^r\sum_{j=0}^{\infty}(j+1)p^j$$
$$\times B\left(a(i+r)+r-1-\frac{(\alpha+1)(r-1)}{\alpha},1+\frac{(\alpha+1)(r+1)}{\alpha}\right)\Bigg]$$

The shannon entropy is defined as $E[-\log f(x)]$. Therefore, by taking the negative $\log$ of the pdf of the GIB, we obatain

$$E(-\log f(x))=-\log\alpha\beta(1-p)+(\alpha+1)E(\log x)+(\beta+1)E(\log(1+x^{-\alpha})).$$
$$+2E\left\{\log\left(1-p(1+x^{-\alpha})\right)\right\}.$$

• for GIB2 we have,

$$I_R(r)=\frac{1}{1-r}\log\left[\left(\alpha\beta(1-p)\right)^r\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}(i+1)^{-1}(-1)^i\frac{\Gamma(2r+j)}{\Gamma(2r)\,j!}p^j\binom{j}{i}\frac{1}{\alpha}\right.$$
$$\left.\times B\left(a(i+r)+r-1-\frac{(\alpha+1)(r-1)}{\alpha},1+\frac{(\alpha+1)(r+1)}{\alpha}\right)\right]$$

The shannon entropy is defined as $E[-\log f(x)]$. Therefore, by taking the negative $\log$ of the pdf of the GIB, we obatain

$$E(-\log f(x))=-\log\alpha\beta(1-p)+(\alpha+1)E(\log x)+(\beta+1)E(\log(1+x^{-\alpha})).$$

6. Simulation

In this section, we assess the finite sample performance of the MLEs of θ = (p; α; β). The results are obtained from generating N samples from GIB1 and GIB2. For each replication, a random sample of size n=50, 100, 200, 300 is drawn from GIB1 and GIB2 and the parameters are then estimated by using the method of maximum likelihood. The GIB1 and GIB2 random generation number were performed by using 10 for GIB1 and 11 for GIB2 . We used three different true parameter values in the data simulation process. The number of replication is set to be N = 10000; Table1 and Table2 show the average MLEs for the three parameters of the GIB1 and GIB2 along with mean squared error. The result reported in Table1 and Table2, from the Tables, we can see that there are convergence and consistency and this emphasize the numerical stability of the MLE method. Also, there is little or no difference between the actual value and simulated value. Comparing the two tables, one can see that, simulating with maximum gave better finite sample behaviour than with the minimum.

Table 1: The average of 10000 MLEs and standard error simulated from GIB1

		AE			SD		
n	(p, α, β)	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$sd(\hat{p})$	$sd(\hat{\alpha})$	$sd(\hat{\beta})$
50	(0.5, 0.5, 2)	0.528	0.561	2.485	0.319	0.557	5.518
	(1.0, 2.0, 1.0)	1.158	2.058	1.977	0.632	1.551	3.775
	(3.0, 2.0, 1.0)	3.559	2.033	1.277	8.694	2.677	1.575
	(3.0, 3.0, 3.0)	3.152	3.034	3.429	1.625	2.045	3.205
100	(0.5, 0.5, 2)	0.544	0.502	2.227	0.271	0.434	5.443
	(1.0, 2.0, 1.0)	1.072	2.033	1.685	0.580	1.429	2.722
	(3.0, 2.0, 1.0)	3.480	2.372	1.112	7.426	2.576	1.452
	(3.0, 3.0, 3.0)	3.068	3.184	3.311	1.037	1.932	3.007
200	(0.5, 0.5, 2)	0.590	0.511	2.579	0.201	0.304	4.475
	(1.0, 2.0, 1.0)	1.032	2.022	1.369	0.448	0.588	1.407
	(3.0, 2.0, 1.0)	3.018	2.332	1.063	3.996	2.275	1.393
	(3.0, 3.0, 3.0)	3.009	3.007	3.012	0.835	1.531	3.005
300	(0.5, 0.5, 2)	0.588	0.501	2.459	0.194	0.205	1.902
	(1.0, 2.0, 1.0)	1.004	2.006	2.264	0.396	0.577	1.203
	(3.0, 2.0, 1.0)	3.070	2.003	1.223	2.311	2.135	0.976
	(3.0, 3.0, 3.0)	3.008	3.110	3.054	0.644	1.623	1.744

Table 2: The average of 10000 MLEs and standard error simulated from GIB2

		AE			SD		
n	(p, α, β)	$\hat{p}$	$\hat{\alpha}$	$\hat{\beta}$	$sd(\hat{p})$	$sd(\hat{\alpha})$	$sd(\hat{\beta})$
50	(0.5, 0.5, 2)	0.582	0.761	4.485	0.291	0.857	5.528
	(1.0, 2.0, 1.0)	1.581	2.085	1.779	0.543	1.451	3.795
	(3.0, 2.0, 1.0)	4.599	4.403	1.257	8.694	2.671	1.475
	(3.0, 3.0, 3.0)	3.152	4.503	3.429	1.625	2.045	3.205
100	(0.5, 0.5, 2)	0.504	0.702	3.722	0.171	0.834	4.473
	(1.0, 2.0, 1.0)	1.072	2.033	1.685	0.358	1.329	2.722
	(3.0, 2.0, 1.0)	4.380	4.372	1.112	6.426	2.572	1.452
	(3.0, 3.0, 3.0)	3.068	3.784	3.311	1.037	1.932	3.007
200	(0.5, 0.5, 2)	0.498	0.711	3.579	0.101	0.204	3.875
	(1.0, 2.0, 1.0)	1.032	2.022	1.369	0.248	0.588	1.407
	(3.0, 2.0, 1.0)	3.358	4.112	1.063	2.996	2.275	1.393
	(3.0, 3.0, 3.0)	2.899	3.779	3.612	0.835	1.931	3.001
300	(0.5, 0.5, 2)	0.488	0.701	2.459	0.094	0.105	1.902
	(1.0, 2.0, 1.0)	1.004	2.006	1.264	0.196	0.577	1.203
	(3.0, 2.0, 1.0)	3.070	4.003	0.999	1.311	2.177	0.976
	(3.0, 3.0, 3.0)	2.778	3.767	3.054	0.644	1.922	1.940

Comparably, we can see that Table 1 which is the average of 10000 MLEs and standard error simulated from GIB1 has better finite sample behavior of the MLEs than Table2 which is the average of 10000 MLEs and standard error simulated from GIB2. This shows that, GIB1 model is better than the GIB2 model.

7. Conclusion

We introduce a double activation approach of lifetime distributions called the geometric inverse burr distribution(GIB), which is obtained by compounding the geometric distribution(GD) and inverse burr (IB) distribution. The ability of the new proposed model is in covering five possible hazard rate function i.e., increasing, decreasing, upside-down bathtub (unimodal), bathtub and increasing-decreasing-increasing shaped. Several properties of the GIB distributions such as moments, maximum likelihood estimation procedure and inference for a large sample, are discussed in this paper. In order to show the flexibility and potentiality of the new distributions, Simulation studies are performed for different parameter values and sample sizes to assess the finite sample behavior of the MLEs. We compare the simulated outcome for the minimum and maximum where we can see from table 2 that, better finite sample behaviour was obtained by using the maximum.

8. Acknowledgements

I would like to thank the Editor and an anonymous referee for their vehement comments in the improvement of this paper.

Appendix A

[i] **Proof.**

$$\text{m(r,n)} = \int_0^\infty x^r f_{\alpha,\beta}(x) F^n_{\alpha,\beta} dx$$
$$= \alpha\beta \int_0^\infty x^r x^{-(\alpha+1)} (1+x^{-(\alpha)})^{-\beta(n+1)-1} dx$$

let $k = (1+x^{-(\alpha)})^{-\beta(n+1)-1}$ after some algebra, we abtain

$$m(r,n) = \frac{\beta}{\beta(n+1)+1} \int_0^1 \left(k^{\frac{1}{\beta(n+1)+1}} \right)^{\frac{r}{\alpha}} k^{\frac{1}{-\beta(n+1)-1}} \left(1 - \left(k^{\frac{1}{\beta(n+1)+1}} \right)^{-\frac{r}{\alpha}} \right) dk$$

Transforming let $u = k^{\frac{1}{\beta(n+1)+1}}$ consequently

$$m(r,n) = aB(a(n+1) + \frac{r}{\alpha}, 1 - \frac{r}{\alpha})$$

[ii] **Proof.**

$$\mu = E[X^r]$$
$$= \int_0^\infty x^r f(x) dx$$
$$= \sum_{i=1}^\infty m_i \alpha\beta \int_0^\infty x^r x^{-(\alpha+1)} (1+x^{-(\alpha)})^{-\beta(n+1)-1} dx$$

after some algebra, we obtain

$$E(X^r) = a \sum_{i=0}^\infty m_i (i+1) B(a(i+1) + \frac{r}{\alpha}, 1 - \frac{r}{\alpha})$$

Appendix B

let $t_i = (1 + x_i^{-\alpha})$

- For GIB1

$$J_{pp} = -\frac{n}{(1-p)^2} - 2\sum_{i=1}^n \frac{(t_i)^{-(\beta)} (1 - pt_i^{-\beta})^2}{}$$

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} + \alpha(\beta+1) \sum_{i=1}^n \frac{\alpha(\alpha+1) t_i \left[x_i^{-\alpha} + x_i^{-(\alpha+1)} \right] - \alpha^2 x_i^{-2(\alpha+1)}}{(t_i)^2}$$
$$-2\beta p \sum_{i=1}^n \frac{(1 - pt_i^{-\beta}) t_i^{-(\beta+1)} [\alpha(\alpha+1) x_i^{-\alpha} + x_i^{-(\alpha+1)}] - \alpha^2 \beta p x_i^{-2(\alpha+1)} t_i^{-2(\beta+1)}}{(1 - pt_i^{-\beta})^2}$$

$$J_{\beta\beta} = -\frac{n}{\beta^2} - 2\alpha p \sum_{i=1}^n \frac{x_i^{-(\alpha+1)} t_i^{-(\beta)}}{(1 - pt_i^{-\beta})}$$

$$J_{p\alpha} = -2\sum_{i=1}^n \frac{\left[\alpha^2 \beta(\beta+1) x_i^{-2(\alpha+1)} t_i^{-(\beta+2)} - \alpha(\alpha+1) t_i^{-(\beta+1)} x_i^{-(\alpha+2)} + x_i^{-(\alpha+1)} \right]}{(1 - pt_i^{-\beta})}$$
$$-2\alpha^2 \beta^2 p \sum_{i=1}^n \frac{x_i^{-2(\alpha+1)} t_i^{-2(\beta+1)}}{(1 - pt_i^{-\beta})^2}$$

$$J_{p\beta}=-2\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)}$$

$$J_{\alpha p}=-2\alpha\beta\sum_{i=1}^{n}\frac{x_i^{-\alpha}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)}-2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}(1-t_i)^{-\beta}}{\left(1-pt_i^{-\beta}\right)^2}$$

$$J_{\alpha\beta}=\alpha\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i}-2\alpha p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)}$$

$$J_{\beta p}=-2\alpha\beta\sum_{i=1}^{n}\frac{px_i^{-(\alpha+1)}t_i^{-(\beta+1)}(1-t_i)^{-\beta}}{\left(1-pt_i^{-\beta}\right)^2}-2\alpha\beta\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-pt_i^{-\beta}\right)}$$

$$J_{\beta\alpha}=\alpha\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i}+2\beta p\sum_{i=1}^{n}\frac{t_i^{-(\beta+1)}\left(x_i^{-(\alpha+1)}-\alpha(\alpha+1)x_i^{-(\alpha+2)}\right)+\alpha^2(\beta+1)x_i^{-2(\alpha+1)}}{\left(1-pt_i^{-\beta}\right)^2}$$

$$+\alpha^2\beta p\sum_{i=1}^{n}\frac{x_i^{-2(\alpha+1)}t_i^{-2(\beta+1)}}{\left(1-pt_i^{-\beta}\right)^2}$$

- For GIB 2

$$J_{pp}=-\frac{n}{(1-p)^2}-2\sum_{i=1}^{n}\frac{(1-t_i)^{-(\beta)})\left(1-p[1-t_i^{-\beta}]\right)^2}{}$$

$$J_{\alpha\alpha}=-\frac{n}{\alpha^2}+\alpha(\beta+1)\sum_{i=1}^{n}\frac{\alpha(\alpha+1)t_i\left[x_i^{-\alpha}+x_i^{-(\alpha+1)}\right]-\alpha^2x_i^{-2(\alpha+1)}}{(t_i)^2}$$

$$-2\beta p\sum_{i=1}^{n}\frac{\left(1-p[1-t_i^{-\beta}]\right)t_i^{-(\beta+1)}[\alpha(\alpha+1)x_i^{-\alpha}+x_i^{-(\alpha+1)}]-\alpha^2\beta px_i^{-2(\alpha+1)}t_i^{-2(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)^2}$$

$$J_{\beta\beta}=-\frac{n}{\beta^2}-2\alpha p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta)}}{\left(1-p[1-t_i^{-\beta}]\right)}$$

$$J_{p\alpha}=-2\sum_{i=1}^{n}\frac{\left[\alpha^2\beta(\beta+1)x_i^{-2(\alpha+1)}t_i^{-(\beta+2)}-\alpha(\alpha+1)t_i^{-(\beta+1)}x_i^{-(\alpha+2)}+x_i^{-(\alpha+1)}\right]}{\left(1-p[1-t_i^{-\beta}]\right)}$$

$$-2\alpha^2\beta^2p\sum_{i=1}^{n}\frac{x_i^{-2(\alpha+1)}t_i^{-2(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)^2}$$

$$J_{p\beta}=-2\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)}$$

$$J_{\alpha p}=-2\alpha\beta\sum_{i=1}^{n}\frac{x_i^{-\alpha}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)}-2\alpha\beta p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}(1-t_i)^{-\beta}}{\left(1-p[1-t_i^{-\beta}]\right)^2}$$

$$J_{\alpha\beta}=\alpha\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i}-2\alpha p\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)}$$

$$J_{\beta p}=-2\alpha\beta\sum_{i=1}^{n}\frac{px_i^{-(\alpha+1)}t_i^{-(\beta+1)}(1-t_i)^{-\beta}}{\left(1-p[1-t_i^{-\beta}]\right)^2}-2\alpha\beta\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}t_i^{-(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)}$$

$$J_{\beta\alpha}=\alpha\sum_{i=1}^{n}\frac{x_i^{-(\alpha+1)}}{t_i}+2\beta p\sum_{i=1}^{n}\frac{t_i^{-(\beta+1)}\left(x_i^{-(\alpha+1)}-\alpha(\alpha+1)x_i^{-(\alpha+2)}\right)+\alpha^2(\beta+1)x_i^{-2(\alpha+1)}}{\left(1-p[1-t_i^{-\beta}]\right)^2}$$

$$+\alpha^2\beta p\sum_{i=1}^{n}\frac{x_i^{-2(\alpha+1)}t_i^{-2(\beta+1)}}{\left(1-p[1-t_i^{-\beta}]\right)^2}$$

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