Common Fixed Point Theorems Weak Compatible in Cone Metric Spaces

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Abstract: ***In this paper we established common fixed point theorems for weakly compatible mappings in cone metric spaces.***

AMS Subject Classification: 47H10, 54H25
Keywords: fixed point, cone metric spaces and weakly compatible mappings.

1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. This contraction principle has further several generalizations in metric spaces as well as in cone metric spaces. Huang and Zhang [1] introduced the concept of cone metric space, where every pair of elements is assigned to an element of a Banach space and defined a partial order on the Banach space with the help of a subset of the Banach space called cone which satisfy certain properties.

2. Preliminary Notes

First, we recall some standard definitions and other results that will be needed in the sequel.

Definition 2.1. Let E be a real Banach space and P be a subset of E. P is called a cone if
(1) P is closed, nonempty and $P \neq \{0\}$;
(2) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.
Given a cone $P \subseteq E$, we define a partial ordering " $\leq$ " in E by $x \leq y$ if $x - y \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x << y$ to denote $y - x \in P^0$, where P^0 stands for the interior of P. We assume cone is solid i. e. that $P^0 \neq \phi$.

Proposition 2.2 [7] : Let P be a cone in a real Banach space E.
(1) If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = 0$.
(2) If $a \in P$ and $a << c$, for all $c \in P^0$ then $a = 0$.

A cone P is called normal if there is constant $K > 0$ such that, for all $x, y \in E$. $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least value of constant K satisfying this inequality is called the normal constant of P.

Definition 2.3 [1]: Let X be a nonempty set and E be a real Banach space. Suppose that the mapping d: X x X $\rightarrow$ E satisfies
(1) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

For examples of cone metric spaces we refer [1, 4].
Henceforth unless otherwise indicated, P is a normal cone in real Banach space
E and " $\leq$ " is partial ordering with respect to P.

Definition 2.4 [1]: Let (X, d) be a cone metric space. Let {xn} be a sequence in X and $x \in X$.
(a) If for every $c \in E$ with $0 \ll c$ (or equivalently $c \in P^0$) there is positive integer n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$ then the sequence $\{x_n\}$ converges to x. We denote this by $x_n \rightarrow x$, as $n \rightarrow \infty$ or $\lim_{n\rightarrow\infty} x_n = x$.

(b) If for every $c \in E$ with $0 \ll c$ there is positive integer n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$ then the sequence $\{x_n\}$ is called a Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 2.5 [1]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \rightarrow 0$ ($n \rightarrow \infty$).

Lemma 2.6 [1]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 0$ ($n,m \rightarrow \infty$).

Definition 2.7: A function $f : P \rightarrow P$ is called subadditive if for all $x, y \in P$, $f(x + y) \leq f(x) + f(y)$.

Definition 2.8: If Y be any partially ordered set with relation " $\leq$ " and $\Psi : Y \rightarrow Y$, we say that Ψ is non decreasing if $x, y \in Y$, $x \leq y \Rightarrow \Psi(x) \leq \Psi(y)$.

Definition 2.9 [2]: Let $\Psi : P \rightarrow P$ be a vector valued function then Ψ is called MS-Altering function if
(a) Ψ is non decreasing, subadditive, continuous and sequentially convergent;
(b) $\Psi(a) = 0$ if and only if $a = 0$.

We replace conditions (a) and (b) by weaker conditions and define cone altering function as follows

Definition 2.10: Let $\Psi : P \to P$ be a vector valued function then Ψ is called cone altering function if
(a) Ψ is non decreasing, subadditive;
(b) $\Psi(a_n) \to 0$ if and only if $a_n \to 0$, for any sequence $\{a_n\}$ in P.

Note that for cone altering function Ψ on normal cone P, $\Psi(a) = 0$ if and only if $a = 0$.

Definition 2.11: Let X be any nonempty set, f, g : X → X be mappings. A point $w \in X$ is called point of coincidence of f and g if there is $x \in X$ such that $fx = gx = w$.

Definition 2.12: Let X be any nonempty set, f, g : X → X be mappings. Pair (f, g) is called weakly compatible if $x \in X$, $fx = gx \Rightarrow fgx = gfx$.

Lemma 2.13: Let (X, d) be a cone metric space and P be a normal cone in a real Banach space E, Ψ is a cone altering function and $k_1, k_2, k > 0$. If $x_n \to x$, $y_n \to y$ in X and $ka \le k_1\Psi[d(x_n, x)] + k_2\Psi[d(y_n, y)]$, then $a = 0$.

Lemma 2.14: Let X be any nonempty set and f, g : X → X be mappings. If (f, g) is weakly compatible pair and have a unique point of coincidence then it is unique common fixed point of f and g.

3. Main Results

Theorem 3.1: Let (X, d) be a cone metric space with normal cone P and let A, S and T be self mappings of X, $\Psi : P \to P$ is cone altering function such that
(3.1.1) $A(X) \subseteq S(X) \cap T(X)$
(3.1.2) the pairs { A, S} and {A, T} be weakly compatible.
(3.1.3i) there exist for all $x, y \in X$

$$\Psi[d(Ax, Ay)] \le a_1\Psi[d(Sx, Ty)]+a_2\Psi[d(Ax, Sx)]+a_3\Psi[d(Ay, Ty)]+a_4\Psi[d(Ax, Ty)] +a_5\Psi[d(Ay, Sx)]$$

where a_i, i = 1, 2, 3, 4, 5 are nonnegative constant such that $a_1+a_2+a_3+a_4+a_5 < 1$. Then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Az = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of A, S and T.

Proof: Let the pairs {A, S} and {A, T} be weakly compatible so there are points $x, y \in X$ such that $Ax = Sx$ and $Ay = Ty$. We claim that $Ax = Ay$. If not, by inequality (3.1.3)

$$\Psi[d(Ax, Ay)] \le a_1\Psi[d(Sx, Ty)]+a_2\Psi[d(Ax, Sx)]+a_3\Psi[d(Ay, Ty)]+a_4\Psi[d(Ax, Ty)] +a_5\Psi[d(Ay, Sx)]$$
$$= a_1\Psi[d(Ax, Ay)]+a_2\Psi[d(Ax, Ax)]+a_3\Psi[d(Ay, Ay)]+a_4\Psi[d(Ax, Ay)] +a_5\Psi[d(Ay, Ax)]$$
$$= (a_1 + a_4 + a_5)\, \Psi[d(Ay, Ax)]$$

since $(a_1 + a_4 + a_5) < 1$ hence by proposition 2.2, we have $\Psi[d(Ax, Ay)] = 0$ i.e. $d(Ax, Ay) = 0$ or $Ax = Ay$.
Therefore $Ax = Sx = Ay = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.1.3) we have $Az = Sz = Ay = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Az = Tz$.

Assume that $w \neq z$. We have

$$\Psi[d(w, z)] = \Psi[d(Aw, Az)]$$
$$\le a_1\Psi[d(Sw, Tz)]+a_2\Psi[d(Aw, Sw)]+a_3\Psi[d(Az, Tz)]+a_4\Psi[d(Aw, Tz)] +a_5\Psi[d(Az, Sw)]$$
$$= a_1\Psi[d(w, z)]+a_2\Psi[d(w, w)]+a_3\Psi[d(z, z)]+a_4\Psi[d(w, z)] +a_5\Psi[d(z, w)]$$
$$= (a_1 + a_4 + a_5)\, \Psi[d(z, w)]$$

since $(a_1 + a_4 + a_5) < 1$ hence by proposition 2.2, we have $\Psi[d(w, z)] = 0$ i.e. $d(w, z) = 0$ or $w = z$ by Lemma 2.14 and z is a unique common fixed point of A, S and T.

Theorem 3.2: Let (X, d) be a cone metric space with normal cone P and let A, B, S and T be self mappings of X, $\Psi : P \to P$ is cone altering function such that
(3.2.1) $A(X) \subseteq S(X)$ and $B(X) \subseteq T(X)$
(3.2.2) the pairs { A, S} and {B, T} be weakly compatible.
(3.2.3) there exist for all $x, y \in X$

$$\Psi[d(Ax, By)] \le a_1\Psi[d(Sx, Ty)]+a_2\Psi[d(Ax, Sx)]+a_3\Psi[d(By, Ty)]+a_4\Psi[d(Ax, Ty)] +a_5\Psi[d(By, Sx)]$$

where a_i, i = 1, 2, 3, 4, 5 are nonnegative constant such that $a_1+a_2+a_3+a_4+a_5 < 1$. Then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs {A, S} and {B, T} be weakly compatible so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (3.2.3)

$$\Psi[d(Ax, By)] \le a_1\Psi[d(Sx, Ty)]+a_2\Psi[d(Ax, Sx)]+a_3\Psi[d(By, Ty)]+a_4\Psi[d(Ax, Ty)] +a_5\Psi[d(By, Sx)]$$
$$= a_1\Psi[d(Ax, By)]+a_2\Psi[d(Ax, Ax)]+a_3\Psi[d(By, By)]+a_4\Psi[d(Ax, By)] +a_5\Psi[d(Ay, Bx)]$$
$$= (a_1 + a_4 + a_5)\, \Psi[d(By, Ax)]$$

since $(a_1 + a_4 + a_5) < 1$ hence by proposition 2.2, we have $\Psi[d(Ax, By)] = 0$ i.e. $d(Ax, By) = 0$ or $Ax = By$.
Therefore $Ax = Sx = By = Ty$. Suppose that there is a another point z such that $Az = Sz$ then by (3.2.3) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.
Assume that $w \neq z$. We have

$$\Psi[d(w, z)] = \Psi[d(Aw, Bz)]$$
$$\le a_1\Psi[d(Sw, Tz)]+a_2\Psi[d(Aw, Sw)]+a_3\Psi[d(Bz, Tz)]+a_4\Psi[d(Aw, Tz)] +a_5\Psi[d(Bz, Sw)]$$
$$= a_1\Psi[d(w, z)]+a_2\Psi[d(w, w)]+a_3\Psi[d(z, z)]+a_4\Psi[d(w, z)]$$

$$+a_5\Psi[d(z, w)]$$
$$= (a_1 + a_4 + a_5)\ \Psi[d(z, w)]$$

since $(a_1 + a_4 + a_5) < 1$ hence by proposition 2.2, we have $\Psi[d(w, z)] = 0$ i.e. $d(w, z) = 0$ or $w = z$ by Lemma 2.14 and z is a unique common fixed point of A, B, S and T.

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