Algebraic Operations: Let $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers.

(i) Addition of Triangular Fuzzy Numbers $\oplus$: $\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1+b_2, c_1+c_2)$

(ii) Multiplication of Triangular Fuzzy Numbers$\otimes$: $\tilde{A} \otimes \tilde{B} = (a_1a_2, b_1b_2, c_1c_2)$; $a_1 \geq 0, a_2 \geq 0$

(iii) Division of Triangular Fuzzy Number $\varnothing$: $\tilde{A} \varnothing \tilde{B} = \left(\frac{a_1}{c_2}, \frac{b_1}{b_2}, \frac{c_1}{a_2}\right)$; $a_1 > 0$, $a_2 > 0$

(iv) Inverse of a Triangular Fuzzy Number: $\tilde{A}^{-1} = (a_1, b_1, c_1)^{-1} = \left(\frac{1}{c_1}, \frac{1}{b_1}, \frac{1}{a_1}\right)$; $a_1 > 0$

A Triangular Fuzzy Number Matrix of order n x m is defined as $A = \left(\tilde{a}_{ij}\right)_{n \times m}$ where $\tilde{a}_{ij}$ is a triangular fuzzy number. The two sets, $X = \{x_1, x_2, x_3, \ldots, x_n\}$ as an object set, and $G = \{u_1, u_2, u_3, \ldots, u_m\}$ as a goal set, can be defined in initial stage. According to the principles of Chang's extent analysis, each object is considered correspondingly, and extent analysis for each of the goal, g_i is executed. It means that it is possible to obtain the values of m extent analyses that can be demonstrated as $M_{g_i}^1, M_{g_i}^2, \ldots, M_{g_i}^m$ i=1,2,…,n, where $M_{g_i}^j$ (j=1,2,…,m) are triangular fuzzy numbers. After identifying initial assumptions, Chang's extent analyses [3], [8], [9] can be examined in four main steps:

Step 1: The value of fuzzy synthetic extent with respect to the ith object is represented as, $F_i = \sum_{j=1}^{m} M_{g_i}^j \otimes \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^j\right]^{-1}$, and fuzzy addition operation of m extent analysis values can be performed for particular matrix such that $\sum_{j=1}^{m} M_{g_i}^j = \left(\sum_{j=1}^{m} l_j, \sum_{j=1}^{m} m_j, \sum_{j=1}^{m} u_j\right)$. Then, the fuzzy addition operation of $M_{g_i}^j$ (j = 1, 2,…., m) values such that $\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^j = \left(\sum_{i=1}^{n} l_i, \sum_{i=1}^{n} m_i, \sum_{i=1}^{n} u_i\right)$ are performed to obtain $\left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{gi}^j\right]^{-1}$. At the end of the Step 1, the inverse of the determined vector can be expressed as follows.

$$\left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^j\right]^{-1} = \left(\frac{1}{\sum_{i=1}^{n} u_i}, \frac{1}{\sum_{i=1}^{n} m_i}, \frac{1}{\sum_{i=1}^{n} l_i}\right)$$

Step 2 : The degree of possibility of $M_1 = (l_1, m_1, u_1) \geq M_2 = (l_2, m_2, u_2)$ is defined as $D(M_1 \geq M_2) = \sup_{x \geq y} [\min(\mu_{M1}(x), \mu_{M2}(x))]$,

When a pair (x, y) exists such that $x \geq y$ and $\mu_{M_1}(x) = \mu_{M_2}(x)$, then we have $D(M_1 \geq M_2) = 1$. Since M_1 and M_2 are convex fuzzy numbers we have that

$D(M_1 \geq M_2) = 1$ if f $m_1 \geq m_2$
$D(M_2 \geq M_1) = hgt(M_1 \cap M_2) = \mu_{M_1}(d)$

Where d is the ordinate of the highest intersection point between $\mu_{M_1}(d)$ and $\mu_{M_2}(d)$. Also the above equation can be equivalently expressed as follows.

$D(M_2 \geq M_1) = hgt(M_1 \cap M_2) = \mu_{M_1}(d)$

1, if $m_2 \geq m_1$,
= 0, if $l_1 \geq u_2$,
$$\frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)} \text{ Otherwise,}$$

Step 3 : From obtaining k(k=1, 2, ….n) convex fuzzy numbers, the degree possibility for a i^th^ convex fuzzy number to be greater than k convex fuzzy numbers M_i (i =1, 2, ….,k) can be defined as follows.

$D(F_i \geq F_k) = D(F_i \geq F_1)$ and $D(F_i \geq F_2) \ldots D(F_i \geq F_k)$
$= D(F_i \geq F_1, F_2, F_3, \ldots F_k)$ with $i \neq k$.

$$d'(A_i) = \min[D(F_i \geq F_1, F_2, F_3, \ldots F_k)] \text{ with } i \neq k.$$

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_n))^T$$

where A_i (i =1, 2, …, n) are n elements.

Step 4: Via normalization, the normalized weight vectors are $W = (d(A_1), d(A_2) \ldots, d(A_n))^T$, where W is a nonfuzzy number that gives weight vectors of an attribute or an alternative over other. Thus we get the original fuzzy AHP decision model with weight vector *W*.

Step 5: Aggregate the relative weights of decision elements to obtain an overall rating for the alternatives. Finally the alternative with highest weight is chosen as the best alternative.

2. Model of the Problem

We define the 4 criteria (C_1, C_2, C_3, C_4) and 4 alternatives (A_1, A_2, A_3, A_4) in order to obtain the best alternative and criterion figure 1.

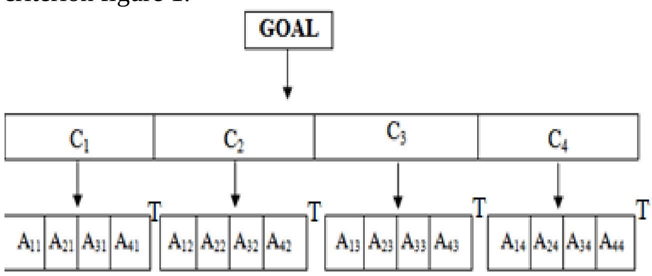

Figure 1: Selection of Best

Construct the Pair wise comparison model for each criterion C_{ij} represents the relative weight of the criterion C_i compared to C_j as follows

Pairwise comparison for the Criteria

Alt./crit.	C_1	C_2	C_3	C_4	
C_1	C_{11}	C_{12}	C_{13}	C_{14}	$C_{1j} = (l_{1j}, m_{1j}, u_{1j})$: j = 1,2,3,4 with $l_{11} = m_{11} = u_{11} = 1$
C_2	C_{21}	C_{22}	C_{23}	C_{24}	$C_{2j} = (l_{2j}, m_{2j}, u_{2j})$: j = 2,3,4 With $l_{22} = m_{22} = u_{22} = 1$
C_3	C_{31}	C_{32}	C_{33}	C_{34}	$C_{3j} = (l_{3j}, m_{3j}, u_{3j})$: j = 3,4 With $l_{33} = m_{33} = u_{33} = 1$
C_4	C_{41}	C_{42}	C_{43}	C_{44}	$C_{4j} = (l_{4j}, m_{4j}, u_{4j})$: j = 4 With $l_{44} = m_{44} = u_{44} = 1$

$C_{i1} = \left(\frac{1}{u_{1i}}, \frac{1}{m_{1i}}, \frac{1}{l_{1i}}\right)$ $C_{i1} = \left(\frac{1}{u_{2i}}, \frac{1}{m_{2i}}, \frac{1}{l_{2i}}\right)$ $C_{i3} = \left(\frac{1}{u_{3i}}, \frac{1}{m_{3i}}, \frac{1}{l_{3i}}\right)$

i = 2, 3, 4 i = 3, 4 i = 4.

Applying all steps of Chang's extent analysis method in the above model, we have the weight vectors of the Criteria C_1, C_2, C_3 and C_4 are obtained as $W_c = (d(C_1), d(C_2), d(C_3)$ $d(C_4))$.

Similarly, Construct the pair wise comparison model for the alternatives A_{11}, A_{21}, A_{31}, A_{41}, with respect to the criterion C_1 and applying all the steps of Chang's extent analysis method in the model. Obtain the weight vectors corresponding to A_{11}, A_{21}, A_{31}, A_{41} respectively

$d(A_{11}), d(A_{21}), d(A_{31}), d(A_{41})$ as

$w_{C_1} = (d(A_{11}), d(A_{21}), d(A_{31}), d(A_{41}))^T$

Similarly, Construct the pair wise comparison model for the alternatives A_{12}, A_{22}, A_{32}, A_{42}, with respect to the criterion C_2 and applying all the steps of Chang's extent analysis method in the model. Obtain the weight vectors corresponding to A_{12}, A_{22}, A_{32}, A_{42} respectively

$d(A_{12}), d(A_{22}), d(A_{32}), d(A_{42})$ as

$w_{C_2} = (d(A_{12}), d(A_{22}), d(A_{32}), d(A_{42}))^T$

Similarly, Construct the pair wise comparison model for the alternatives A_{13}, A_{23}, A_{33}, A_{43}, with respect to the criterion C_3 and applying all the steps of Chang's extent analysis method in the model. Obtain the weight vectors corresponding to A_{13}, A_{23}, A_{33}, A_{43} respectively

$d(A_{13}), d(A_{23}), d(A_{33}), d(A_{43})$ as

$w_{C_3} = (d(A_{13}), d(A_{23}), d(A_{33}), d(A_{43}))^T$

Similarly, Construct the pair wise comparison matrix for the alternatives A_{14}, A_{24}, A_{34}, A_{44}, with respect to the criterion C_4 and applying all the steps of Chang's extent analysis method in the model. Obtain the weight vectors corresponding to A_{14}, A_{24}, A_{34}, A_{44} respectively

$d(A_{14}), d(A_{24}), d(A_{34}), d(A_{44})$ as

$w_{C_4} = (d(A_{14}), d(A_{24}), d(A_{34}), d(A_{44}))^T$

Thus we get the original fuzzy AHP decision model and their weight vectors, using above weight vectors of Criteria and alternatives

Fuzzy AHP model

Alternative /criterion	C_1	C_2	C_3	C_4	Final Weight Vector
Weight Vectors of Criteria	$d(C_1)$	$d(C_2)$	$d(C_3)$	$d(C_4)$	A^2_{AHP}
A_1	$d(A_{11})$	$d(A_{12})$	$d(A_{13})$	$d(A_{14})$	A^1_{AHP}
A_2	$d(A_{21})$	$d(A_{22})$	$d(A_{23})$	$d(A_{24})$	A^2_{AHP}
A_3	$d(A_{31})$	$d(A_{32})$	$d(A_{33})$	$d(A_{34})$	A^3_{AHP}
A_4	$d(A_{41})$	$d(A_{42})$	$d(A_{43})$	$d(A_{44})$	A^4_{AHP}

Ideal fuzzy AHP decision model using original fuzzy AHP decision model and their weight vector are obtained as follows.

Ideal AHP model

Alternative /criterion	C_1	C_2	C_3	C_4	Final Weight Vector
Weight Vectors of Criteria	$d(C_1)$	$d(C_2)$	$d(C_3)$	$d(C_4)$	
A_1	$d(IA_{11})$	$d(IA_{12})$	$d(IA_{13})$	$d(IA_{14})$	IA_1
A_2	$d(IA_{21})$	$d(IA_{22})$	$d(IA_{23})$	$d(IA_{24})$	IA_2
A_3	$d(IA_{31})$	$d(IA_{32})$	$d(IA_{33})$	$d(IA_{34})$	IA_3
A_4	$d(IA_{41})$	$d(IA_{42})$	$d(IA_{43})$	$d(IA_{44})$	IA_4

After normalization, we have the ranking the alternatives.

It can also be extended to find the final alternative weight vectors for each alternative from the original fuzzy AHP decision model. It can by obtained from the following ways.

$$MA_i = \sum_{j=1}^{4} d(C_j)\,[d(C_j) + d(A_{ij})] \text{ for all}$$

Thus the moderate fuzzy AHP [7] decision matrix is obtained as follows.

Moderate AHP Model

	C_1	C_2	C_3	C_4	Final Weight Vector
Weight Vectors of Criteria	$d(C_1)$	$d(C_2)$	$d(C_3)$	$d(C_4)$	
A_1	$d(MA_{11})$	$d(MA_{12})$	$d(MA_{13})$	$d(MA_{14})$	MA_1
A_2	$d(MA_{21})$	$d(MA_{22})$	$d(MA_{23})$	$d(MA_{24})$	MA_2
A_3	$d(MA_{31})$	$d(MA_{32})$	$d(MA_{33})$	$d(MA_{34})$	MA_3
A_4	$d(MA_{41})$	$d(MA_{42})$	$d(MA_{43})$	$d(MA_{44})$	MA_4

After normalization, we have the ranking the alternatives. Finally we have the same ranking for original fuzzy AHP decision model, Ideal fuzzy AHP decision model and moderate fuzzy AHP decision model, even though different the value of the final alternative weight vectors of respective alternatives.

3. Numerical Example

Decision makers determine goal, Criteria and alternative of the problem in a hierarchical form. This hierarchy has to give the all details of the information on the structure in order to give lack less of the problem. Decision makers are required to compare each factor in the hierarchy. The evaluation of the fuzzy scale used by the decision makes shown in the table 1.

Table 1: Fuzzy AHP scale

S. No.	*Definition*	*Triangular Fuzzy Number*	*Reciprocal Fuzzy Number*
1.	Equally importance	(1,1,1)	(1,1,1)
2.	Weakly importance	(1,1,3)	(1/3, 1,1)
3.	Moderately importance	(1,3,3)	(1/3, 1/3,1)
4.	Strongly importance	(1,3,5)	(1/5, 1/3, 1)
5.	Very strongly importance	(3,5,7)	(1/7, 1/5, 1/3)
6.	Extremely importance	(5,7,9)	(1/9, 1/7, 1/5)

Pairwise comparison of Criteria

Alternative /criterion	C_1	C_2	C_3	C_4
C_1	(1,1,1)	(1,1,3)	(1,3,3)	(3,5,7)
C_2	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,3,5)	(3,5,7)
C_3	$(\frac{1}{5},\frac{1}{3},1)$	$(\frac{1}{5},\frac{1}{3},1)$	(1,1,1)	(1,3,5)
C_4	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{5},\frac{1}{3},1)$	(1,1,1)

The normalized weight vector for criteria are calculated as w_e = (0.3704, 0.3704, 02386, 0.0206)

Pairwise comparison matrix for alternatives with respect to C_1

C_1	A_{11}	A_{21}	A_{31}	A_{41}
A_{11}	(1,1,1)	(1,1,3)	$(\frac{1}{5},\frac{1}{3},1)$	(3,5,7)
A_{21}	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,1,3)	(5,7,9)
A_{31}	(1, 3, 5)	$(\frac{1}{3}, 1, 1)$	(1,1,1)	$(\frac{1}{9},\frac{1}{7},\frac{1}{5})$
A_{41}	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{9},\frac{1}{7},\frac{1}{5})$	(5,7,9)	(1,1,1)

The normalized weight vectors with respect to C_1 are calculated as w_{C_1} = (0.2565, 0.3124, 0.1627, 0.2684)^T

Pairwise comparison model for alternatives with respect to C_2.

2	A_{12}	A_{22}	A_{32}	A_{42}
A_{12}	(1,1,1)	(1,1,3)	(3,5,7)	$(\frac{1}{5},\frac{1}{3},1)$
A_{22}	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,1,3)	$(\frac{1}{9},\frac{1}{7},\frac{1}{5})$
A_{32}	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,3,5)
A_{42}	(1, 3, 5)	(5,7,9)	$(\frac{1}{5},\frac{1}{3},1)$	(1,1,1)

The normalized w_{C_2} = (0.3005, 0.1114, 0.2011, 0.3869)^T

Pairwise comparison model for alternatives with respect to C_3.

C_3	A_{13}	A_{23}	A_{33}	A_{43}
A_{13}	(1,1,1)	(1,1,3)	(1,3,5)	$(\frac{1}{9},\frac{1}{7},\frac{1}{5})$
A_{23}	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,3,3)	(1,3,5)
A_{33}	$(\frac{1}{5},\frac{1}{3},1)$	$(\frac{1}{3},\frac{1}{3},1)$	(1,1,1)	$(\frac{1}{3},\frac{1}{3},1)$
A_{43}	(5,7,9)	$(\frac{1}{5},\frac{1}{3},1)$	(1,3,3)	(1,1,1)

The normalized weight vectors with respect to C_3 are calculated as w_{C_3} = (0.2457, 0.3046, 0.0601, 0.3896)^T

Pairwise comparison model for alternatives with respect to C_4.

C_4	A_{14}	A_{24}	A_{34}	A_{44}
A_{14}	(1,1,1)	(1,1,3)	(3,5,7)	(3,5,7)
A_{24}	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(1,1,3)	(1,3,3)
A_{34}	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{3}, 1, 1)$	(1,1,1)	(5,7,9)
A_{44}	$(\frac{1}{7},\frac{1}{5},\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},1)$	$(\frac{1}{9},\frac{1}{7},\frac{1}{5})$	(1,1,1)

The normalized weight vectors with respect to C_4 are calculated as

w_{C_4} = (0.4343, 0.2191, 0.3466, 0.0000)^T

From weight vectors of criteria and alternatives, we have fuzzy AHP decision models as follows.

Thus original fuzzy AHP decision model

Alternative /criterion	C_1	C_2	C_3	C_4	Final weight Vector	Ranking
Crit Weight	0.3704	0.3704	0.2386	0.0206		
A_1	0.2565	0.3005	0.2457	0.4343	0.2738	2
A_2	0.3124	0.1114	0.3046	0.2191	0.2342	3
A_3	0.1627	0.2011	0.0601	0.3466	0.1562	4
A_4	0.2684	0.3869	0.3896	0.0000	0.3357	1

Ideal mode fuzzy AHP decision model

Alternative /criterion	C_1	C_2	C_3	C_4	Final weight Vector	Normalization	Ranking
Crit Weight	0.3704	0.3704	0.2386	0.0206			
A_1	0.8211	0.7767	0.6306	1.0000	0.7629	0.2722	2
A_2	1.0000	0.2870	0.7818	0.5045	0.6736	0.2404	3
A_3	0.5208	0.5198	0.1543	0.7981	0.4386	0.1565	4
A_4	0.8592	1.0000	1.0000	0.0000	0.9272	0.3309	1

Moderate fuzzy AHP decision model

Alternative /criterion	C_1	C_2	C_3	C_4	**Final weight Vector**	Normalization	Ranking
Crit Weight	0.3704	0.3704	0.2386	0.0206			
A_1	0.2322	0.2485	0.1156	0.0094	0.6057	0.2603	2
A_2	0.2529	0.1785	0.1296	0.0049	0.5659	0.2432	3
A_3	0.1975	0.2117	0.0713	0.0076	0.4881	0.2098	4
A_4	0.2366	0.2805	0.1499	0.0000	0.6670	0.2866	1

Therefore, the best selection is A_4 followed by A_1, A_1 is followed by A_2 and A_2 is followed by A_3. Finally we observe that the original fuzzy AHP, the ideal fuzzy AHP and the moderate fuzzy AHP have the same ranking for the said 4 alternatives, even though they assigned different final weight vectors for these alternatives.

4. Conclusion

The fuzzy AHP is used for ranking with weight vectors of pairwise comparison matrices. It provides an effective solution for solving MCDM problem. We can involve any relative importance of criteria and that of alternatives in the moderate fuzzy AHP. Also moderate fuzzy AHP allows for a sensitivity analysis in term of the relative priorities, by adjusting the ranking values. Application of the moderate fuzzy AHP of the MCDM can be discussed in further research proposals. The numerical problem shows the proposed fuzzy analysis and its applicability in providing a valuable decision support.

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