

On Ternary Quadratic Homogeneous Diophantine Equation $5x^2 + 5y^2 - 9xy = 20z^2$

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Abstract- Five different methods of the non-zero integral solutions of the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 - 9xy = 20z^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: Ternary Quadratic, homogenous cone, integer solutions, special numbers.

2010 Mathematics Subject Classification: 11D09

Notations Used

$t_{m,n}$ - Polygonal number of rank n with size m

P_n - Pronic number of rank n

g_a - Gnomonic number of rank a

P_n^m - Pyramidal number of rank n with size m

1. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting five different methods of the non-zero integral solutions the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 - 9xy = 20z^2$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Method of Analysis

The ternary quadratic homogeneous Diophantine equation to be solved is

$$5x^2 + 5y^2 - 9xy = 20z^2 \quad (1)$$

Introducing the linear transformation

$$x = u + v, y = u - v, u \neq v \neq 0 \quad (2)$$

in (1), it reduces to

$$u^2 + 19v^2 = 20z^2 \quad (3)$$

We discuss below five different methods of non-zero distinct integer solutions of (3)

2.1 Method: 1

$$\text{Let } z = a^2 + 19b^2, \text{ where } a, b \neq 0 \quad (4)$$

Write 20 as

$$20 = (1 + i\sqrt{19})(1 - i\sqrt{19}) \quad (5)$$

Substituting (4) & (5) in (3) and employing the method of factorization, we get

$$(u + i\sqrt{19}v) = (1 + i\sqrt{19})(a + i\sqrt{19}b)^2$$

On equating real and imaginary parts, we get

$$u = u(a, b) = a^2 - 19b^2 - 38ab$$

$$v = v(a, b) = a^2 - 19b^2 + 2ab$$

In view of (2), we get the non-zero integral solutions of (1) in two parameters.

$$x = x(a, b) = 2a^2 - 38b^2 - 36ab$$

$$y = y(a, b) = -40ab$$

Properties

- $x(A, A) + 74t_{4,A} = 0$
- $x(A, A+1) + 72t_{4,A} + g_{56A} + 39 = 0$
- $y(A, A+1) + 40P_A = 0$
- $y(A, A) + 40t_{4,A} = 0$
- $z(A, B) - t_{4,A} - 19t_{4,B} = 0$
- $z(A(A+1), B(B+1)) - P_A^2 - 19P_B^2 = 0$
- $y(A(A+1), A) + 80P_A^5 = 0$

2.1 Method: 2

Write (3) as

$$19v^2 = 20z^2 - u^2 \quad (6)$$

Write 19 as

$$19 = (\sqrt{20} + 1)(\sqrt{20} - 1) \quad (7)$$

$$\text{Let } v = 20a^2 - b^2, \text{ where } a, b > 0 \quad (8)$$

Using (7) in (6), we get

$$20z^2 - u^2 = (\sqrt{20} + 1)(\sqrt{20} - 1)(20a^2 - b^2)^2$$

On employing the method of factorization & equating rational & irrational parts, we get

$$z = z(a, b) = 20a^2 + b^2 + 2ab$$

$$u = u(a, b) = 20a^2 + b^2 + 40ab$$

Substituting u & v in (2), we get the non-zero integral solutions of (1) in two parameters

$$z = z(a, b) = 40a^2 + 40ab$$

$$y(a, b) = 2b^2 + 40ab$$

Properties

- $x(A, A+1) - 40t_{4,A} - 40P_A = 0$
- $y(A(A+1), 1) - 40P_A = 2$
- $z(B, B+1) - 23t_{4,B} - g_{2B} = 0$
- $x(A(A+1), 1) - 40P_A^2 - 40P_A = 0$
- $y(A, 1) - 2t_{44,A} + 42t_{4,A} \equiv 0 \pmod{2}$
- $x(A, A(A+1)) - 40t_{4,A} - 80P_A^5 = 0$
- $z(A, 1) - 2t_{22,A} + g_{8A} = 0$

2.3 Method: 3

Write (3) as

$$1 \cdot u^2 = 20z^2 - 19v^2 \quad (9)$$

$$\begin{aligned} \text{Assume } u &= u(a,b) = 20a^2 - 19b^2 \\ &= (\sqrt{20}a + \sqrt{19}b)(\sqrt{20}a - \sqrt{19}b) \end{aligned} \quad (10)$$

Write 1 as

$$1 = (\sqrt{20} + \sqrt{19})(\sqrt{20} - \sqrt{19}) \quad (11)$$

Using (10) & (11) in (9), we get

$$\begin{aligned} 20z^2 - 19v^2 &= (\sqrt{20} + \sqrt{19})(\sqrt{20} - \sqrt{19})(\sqrt{20}a + \sqrt{19}b)^2(\sqrt{20}a - \sqrt{19}b)^2 \end{aligned}$$

On employing the method of factorization and equating the rational & irrational parts, we get

$$z = z(a, b) = 20a^2 + 19b^2 + 38ab$$

$$v = v(a, b) = -20a^2 + 19b^2 + 40ab$$

Substituting u & v in (2), we get

$$x = x(a, b) = 40a^2 + 40ab$$

$$y = y(a, b) = -38b^2 - 40ab$$

Properties

1. $x(A, A+1) - 40t_{4,A} - 40P_A = 0$
2. $y(A, 1) + g_{20A} + 39 = 0$
3. $z(A, A) - 77t_{4,A} = 0$
4. $z(1, B) - t_{4,B} - g_{19B} \equiv 1 \pmod{2}$
5. $x(A, A(A+1)) - 40t_{4,A} - 80P_A^5 = 0$
6. $y(A, A) + 2t_{80,A} - g_{38A} = 1$
7. $z(A(A+1), 1) - 20P_A^2 - 30P_A = 19$

2.4 Method: 4

Write (3) as

$$\begin{aligned} u^2 + 19v^2 &= 19z^2 + z^2 \\ u^2 - z^2 &= 19z^2 - 19v^2 \\ (u+z)(u-z) &= 19(z+v)(z-v) \end{aligned} \quad (12)$$

Write (12) as

$$\frac{u+z}{z-v} = \frac{19(z+v)}{u-z} = \frac{A}{B}, B \neq 0,$$

which is equivalent to the system of equations in three unknowns

$$Bu + Z(B - A) + AV = 0$$

$$-Au + Z(A + 19B) + 19BV = 0$$

By cross-multiplication, we get

$$u = u(A, B) = -A^2 + 19B^2 - 38AB$$

$$v = v(A, B) = -A^2 + 19B^2 + 2AB$$

$$z = z(A, B) = -A^2 - 19B^2$$

Substituting the values of u & v in (2), we get

$$x = x(A, B) = -2A^2 + 38B^2 - 36AB$$

$$y = y(A, B) = -40AB$$

$$z = z(A, B) = -A^2 - 19B^2$$

Properties

1. $x(A, 1) + 2t_{40,A} - g_{18A} \equiv 0 \pmod{3}$
2. $z(A, A) + 2t_{22,A} - g_{9A} = 1$
3. $y(A, A+1) + 40P_A = 0$
4. $x(A(A+1), 1) + 2P_A^2 + 36P_A \equiv 0 \pmod{2}$
5. $z(A, B) + t_{4,A} + 19t_{4,B} = 0$
6. $y(A, A(A+1)) + 80P_A^5 = 0$

$$7. \quad x(A, A(A+1)) + 2t_{4,A} - 38P_A^2 + 72P_A^5 = 0$$

2.5 Method: 5

Write 20 as

$$20 = \frac{(31 + i\sqrt{19})(31 - i\sqrt{19})}{49} \quad (13)$$

$$\text{Assume } z = a^2 + 19b^2 \quad (14)$$

Substituting equation (13) & (14) in (3) and employing the method of factorization, we get

$$(u + i\sqrt{19}v) = \frac{1}{7}(9 + i\sqrt{19})(9 + i\sqrt{19}b)^2$$

Equating real and imaginary parts, we get

$$u = u(a, b) = \frac{1}{7}(31a^2 - 5891b^2 - 38ab)$$

$$v = v(a, b) = \frac{1}{7}(a^2 - 19b^2 + 62ab)$$

In view of (2), we get

$$x = x(a, b) = \frac{1}{7}[32a^2 - 608b^2 + 24ab] \quad (15)$$

$$y = y(a, b) = \frac{1}{7}(30a^2 - 570b^2 - 100ab) \quad (16)$$

Choose a, b suitably so that x, y, and z are in integers.

Putting a = 7A, b = 7B in (15) & (16), we get the non-zero

integral solutions of (1) in two parameters

$$x = x(A, B) = 224A^2 - 4256B^2 + 168AB$$

$$y = y(A, B) = 210A^2 - 3990B^2 - 700AB$$

$$z = z(A, B) = 49A^2 + 931B^2$$

Properties

1. $z(A, B) - 49t_{4,A} - 931t_{4,B} = 0$
2. $z(A(A+1), B) - 49P_A^2 - 931t_{4,B} = 0$
3. $y(1, A(A+1)) - 3990P_A^2 + 700P_A \equiv 0 \pmod{7}$
4. $x(A, A) + 3864t_{4,A} = 0$
5. $x(A(A+1), B(B+1)) - 49P_A^2 - 931P_B^2 = 0$
6. $y(1, B) + 3990t_{4,B} + g_{350B} \equiv 0 \pmod{3}$
7. $z(1, A(A+1)) - 931P_A^2 \equiv 0 \pmod{7}$
8. $x(B(B+1), 1) - 224P_B^2 - 168P_B \equiv 0 \pmod{2}$

3. Conclusion

In this work, we have observed five different patterns of the non-zero integral solutions of the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 - 9xy = 20z^2$ and relations between solutions and special numbers are also obtained. One may research for any other patterns of this equation and their corresponding properties.

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