Integral Solutions of the Homogeneous Biquadratic Diophantine Equations with Five Unknowns

\((X^2 - Y^2) (3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2\)

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Abstract: Four different patterns are used to find non-zero distinct integral solutions for the homogeneous biquadratic Diophantine equations \((X^2 - Y^2) (3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2\). Different types of properties are exposed in every pattern with polygonal, nasty, square and cubic numbers.

Keywords: Homogeneous biquadratic, integral solutions, special numbers


1. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting four different methods of the non-zero integral solutions of the homogeneous Biquadratic Diophantine equations with five unknowns \((X^2 - Y^2) (3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2\). Further, some elegant properties among the special numbers and the solutions are observed.

2. Method of Analysis

The homogeneous Biquadratic Diophantine equations to be solved is

\((X^2 - Y^2) (3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2\)  \(\text{(1)}\)

Introducing the linear transformations

\[ X = u + v, \quad Y = u - v, \quad Z = 2u + v, \quad W = 2u - v, \quad \text{where} \quad u \neq v \neq 0 \]

\(\text{in (1)}, \quad \text{we get} \]

\[ u^2 + 2v^2 = 6T^2 \]  \(\text{(2)}\)

Assume \(T(a, b) = a^2 + 2b^2\), where \((a, b \neq 0)\)

We present below four different patterns of non-zero distinct integer solutions to (1).

### 2.1 Pattern 1:

Take 6 as

\[ 6 = (2 + i\sqrt{3}) (2 - i\sqrt{3}) \]  \(\text{(5)}\)

Substituting (4) and (5) in (3) and applying the method of factorization, we get

\[ (\alpha + i\sqrt{2}v) = (2 + i\sqrt{3}) (\alpha + i\sqrt{2}b)^2 \]

Equating real and imaginary parts, we get

\[ u = 2a^2 - 4b^2 - 4ab \]  \(\text{(6)}\)

\[ v = a^2 - 2b^2 + 4ab \]  \(\text{(7)}\)

Putting (6) and (7) in (2), we get non-zero distinct integer valued for \(x, y, z, w\) and satisfying (1) are given below

\[ X = X(a, b) = 3a^2 - 6b^2 \]  \(\text{(8)}\)

\[ Y = Y(a, b) = a^2 - 2b^2 - 8ab \]  \(\text{(9)}\)

\[ Z = Z(a, b) = 5a^2 - 10b^2 - 4ab \]  \(\text{(10)}\)

\[ W = W(a, b) = 3a^2 - 6b^2 - 12ab \]  \(\text{(11)}\)

The equations (8) to (11) and (4) give non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. \(X(A, A (A+1)) + 24T_{3A}^2 = 3T_{4A}\)
2. \(Y(A (A+1), A+2) - 4T_{3A}^2 + T_{6A} + 48P_A^3 \equiv -8 (\text{mod} 9)\)
3. \(X(A, A+1) - 3Y(A, A+1) = 8P_A\)
4. \(5Y(A, 1) - 2Z(A, 1) + g_i(a+b+1) = 0\)
5. \(8T(A, A) \text{ is a Nasty number}\)
6. \(-6Y(A, A) \text{ is a Nasty number}\)
7. \(-2 \{w(A, A)\} \text{ is a Nasty number}\)

### 2.2 Pattern 2

Instead of (5), Take 6 as

\[ 6 = (-2 + i\sqrt{3}) (-2 - i\sqrt{3}) \]  \(\text{(12)}\)

We do the same procedure as in pattern:1 and from (2) we get

\[ X = X(a, b) = -a^2 + 2b^2 - 8ab \]  \(\text{(13)}\)

\[ Y = Y(a, b) = -3a^2 + 6b^2 \]  \(\text{(14)}\)

\[ Z = Z(a, b) = -3a^2 + 6b^2 - 12ab \]  \(\text{(15)}\)
\( W = W(a, b) = -5a^2 + 10b^2 - 4ab \quad (16) \)

The equations (13) to (16) and (4) give non-zero distinct integral solutions of (1) in two parameters.

**Properties:**

1. \( X(A, A + 1) + T_{4,A} + 16 P_A^5 = 8 T_{3,A}^2 \)
2. \( X(A, 1) - 3Y(A, 1) + T(A, 1) - T_{4,A} \equiv 2 \pmod{24} \)
3. \( 5Y(A, 1) - Z(A, 1) + T(A, 1) - T_{4,A} \equiv 2 \pmod{36} \)
4. \( X(A, 1) - W(A, 1) + T(A, 1) - 12Pr_A = 2 \pmod{11} \)
5. \( [W(A, A)] \) is a Square number
6. \( 8 \{X(A, A) + Y(A, A)\} \) is a Nasty number
7. \( 5 \{Z(A, A) + W(A, A)\} \) is a Cubic number
8. \( 3 \) \{Z(A, A) + W(A, A)\} is a Nasty number

2.4 Pattern:4

Rewrite (1) as

\[ 6T^2 - w^2 = 2v^2 \quad (28) \]

Consider 2 as

\[ 2 = (\sqrt{6} + 2)(\sqrt{6} - 2) \quad (29) \]

Assume \( v(a, b) = 6a^2 - b^2; a, b \neq 0 \) \quad (30)

Properties:

1. \( X(A, A + 1) - T(A, A + 1) + 12Pr_A = 12T_{4,A} \)
2. \( 5Y(1, B) + T(1, B) - 12Pr_B = 12T_{4,B} \)
3. \( X(1, B) + T(1, B) - T_{4,B} \equiv 0 \pmod{3} \)
4. \( [Y(A, A) + T(A, A)] \) is a Nasty number
5. \( 2 \{X(A, A) + Y(A, A)\} \) is a Square number
6. \( 3 \) \{Z(A, A) + W(A, A)\} is a Nasty number

3. Conclusion

In this work, we have observed four different patterns of the non-zero integer solutions of the homogeneous Biquadratic Diophantine equation \( (X^2 - Y^2) (3X^2 + Y^2 - 2XY) = 12(\sqrt{2} - W^2)T^2 \) and relations between solutions and special numbers are also obtained. One may research for any other patterns of this equation and their corresponding properties.

**References**


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