On a New Subclass of Meromorphic Univalent Functions Defined by Integral Operator

Waggas Galib Atshan[1], Safa Ehab Mohammed[2]

Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya – Iraq

Abstract: ***In this paper we introduce a new subclass of meromorphic univalent functions in the punctured unit disk with integral operator. We obtain coefficient inequality, Closure Theorem, convex combination, distortion bound, Partial sums and neighborhood property.***

Keywords: Meromorphic univalent functions, integral operator, Closure Theorem, convex combination, distortion bound, Partial sums and neighborhood property

1. Introduction

Let $\mathcal{A}_1^*$ be the class of functions f of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k,$$

which are analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

Let Σ be the subclass of $\mathcal{A}_1^*$, consisting of functions of the form:

$$f(z) = z^{-1} - \sum_{k=1}^{\infty} a_k z^k \;, \quad (a_k \geq 0, k \in \mathbb{N} = \{1,2, \dots\}), \qquad (1.1)$$

which are analytic and meromorphic univalent in U^*.

A function $f \in \Sigma$ is said to be meromorphicstarlike function of order $\rho(0 \leq \rho < 1)$ if $-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$, $(z \in U = U^* \cup \{0\}, 0 \leq \rho < 1)$. (1.2)
The class of all such functions is defined by$\Sigma^*(\rho)$.

A function $f \in \Sigma$is said to be meromorphic convex function of order $\rho(0 \leq \rho < 1)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (z \in U = U^* \cup \{0\}, 0 \leq \rho < 1). \qquad (1.3)$$

Definition (1.1)[6]: Analogous to the operators defined by Jung, Kim, and Srivastava [6] on the normalized analytic functions, by [1] define the following integral operator

$$P_\beta^\alpha : \Sigma \longrightarrow \Sigma$$

$$P_\beta^\alpha = P_\beta^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{z^{p+1}} \int_0^z t^\beta (log\frac{z}{t})^{\alpha-1} f(t)dt, (\alpha, \beta > 0; z \in U), \qquad (1.4)$$

where $\Gamma(\alpha)$ is the familiar Gamma function.
Using the integral representation of the Gamma and Beta for $f(z) \in \Sigma$, given by (1.1) we have

$$P_\beta^\alpha f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{\beta}{n+\beta+1}\right)^\alpha a_n z^n, (\alpha > 0, \beta > 0) \qquad (1.5)$$

Lemma (1.1)[2]: Let $\alpha \geq 0$ and w=-(u+iv) is complex number,Then $Re(w) > \alpha$ if and only if $|w - (1+\alpha)| \leq |w + (1-\alpha)|$.

Lemma (1.2) [4]: If $w = u + iv$ is a complex number and δ, k are real numbers, then $-Re(w) \geq k|w+1| + \delta$ if and only if

$$-Re\left(w\left(1 + ke^{i\theta}\right)\right) + ke^{i\theta} \geq \delta, \qquad -\pi \leq \theta \leq \pi.$$

Definition (1.2):A function $f \in \Sigma$ is said to be in the class A$(\delta, \lambda, k, \alpha, \beta)$if

$$-Re\left\{\frac{z\left(P_\beta^\alpha f(z)\right)'}{P_\beta^\alpha f(z)} + \frac{z^2\left(P_\beta^\alpha f(z)\right)''}{P_\beta^\alpha f(z)} + \frac{\frac{\lambda}{3}z^3\left(P_\beta^\alpha f(z)\right)'''}{P_\beta^\alpha f(z)} + \delta\right\} > k\left|\frac{z\left(P_\beta^\alpha f(z)\right)'}{P_\beta^\alpha f(z)} + \frac{z^2\left(P_\beta^\alpha f(z)\right)''}{P_\beta^\alpha f(z)} + \frac{\frac{\lambda}{3}z^3\left(P_\beta^\alpha f(z)\right)'''}{P_\beta^\alpha f(z)} + 1\right|,$$

$$\left(0 \leq \delta < 1, k \geq 0, 0 \leq \lambda < \frac{1}{3}; \alpha, \beta > 0\right) \qquad (1.6)$$

The first theorem gives a necessary and sufficient condition for a function f to be in the class A$(\delta, \lambda, k, \alpha, \beta)$.

2. Coefficient Bounds

Theorem (2.1): Let $f \in \Sigma$. Then $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$if and only if

$$\sum_{n=1}^{\infty}\left[n(1+k)\left(n + \frac{\lambda}{3}(n-1)(n-2)\right) + k + \delta\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_n \leq 2k(1-\lambda) - 2\lambda + 1 + \delta. \qquad (2.1)$$

Proof : Let $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ and $|z| = 1$, Then by Lemma (1.2) we have

$$-Re\left\{\frac{z\left(P_\beta^\alpha f(z)\right)'}{P_\beta^\alpha f(z)}+\frac{z^2\left(P_\beta^\alpha f(z)\right)''}{P_\beta^\alpha f(z)}+\frac{\frac{\lambda}{3}z^3\left(P_\beta^\alpha f(z)\right)'''}{P_\beta^\alpha f(z)}\left(1+ke^{i\theta}\right)+ke^{i\theta}\right\}\geq\delta. \qquad (2.2)$$

Let

$$A(z)=-\left(z\left(P_\beta^\alpha f(z)\right)'+z^2\left(P_\beta^\alpha f(z)\right)''+\frac{\lambda}{3}z^3\left(P_\beta^\alpha f(z)\right)'''\right)\left(1+ke^{i\theta}\right)-P_\beta^\alpha f(z)ke^{i\theta}$$

and

$B(z)=P_\beta^\alpha f(z)$.

The equation (2.2) is equivalent to

$$Re\left\{\frac{A(z)}{B(z)}\right\}\geq\delta.$$

In view of Lemma (1.1), we only need to prove that

$$|A(z)+(1-\delta)B(z)|-|A(z)-(1+\delta)B(z)|\geq 0.$$

Therefore

$|A(z)+(1-\delta)B(z)|=$

$$=\left|-\left(2ke^{i\theta}(1-\lambda)-2\lambda+\delta\right)z^{-1}+\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)\left(1+ke^{i\theta}\right)+ke^{i\theta}-(1-\delta)\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n\right|$$

$$\geq(2k(1-\lambda)-2\lambda+\delta)|z^{-1}|-\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k-(1-\delta)\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n,$$

and

$|A(z)-(1+\delta)B(z)|=$

$$=\left|(-2+2\lambda)-\delta+2ke^{i\theta}(-1+\lambda)z^{-1}+\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)\left(1+ke^{i\theta}\right)+ke^{i\theta}+(1+\delta)\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n\right|$$

$$\leq(-2+2\lambda)-\delta+2k(-1+\lambda)|z^{-1}|+\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k+(1+\delta)\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_n|z^n|,$$

then

$$|A(z)+(1-\delta)B(z)|-|A(z)-(1+\delta)B(z)|$$
$$\geq(2k(1-\lambda)-2\lambda+\delta)|z^{-1}|-\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)-k-(1-\delta)\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n$$
$$+2-2\lambda+\delta+2k(1-\lambda)|z^{-1}|-\sum_{n=1}^{\infty}\left[n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k+\right.$$

(1+δ)βn+β+1αanzn,

which is equivalent to

$$\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_n\leq 2k(1-\lambda)-2\lambda+1+\delta.$$

Conversely, suppose that (2.1) hold true, then, we must show

$$-Re\left\{\frac{z\left(P_\beta^\alpha f(z)\right)'}{P_\beta^\alpha f(z)}+\frac{z^2\left(P_\beta^\alpha f(z)\right)''}{P_\beta^\alpha f(z)}+\frac{\frac{\lambda}{3}z^3\left(P_\beta^\alpha f(z)\right)'''}{P_\beta^\alpha f(z)}\left(1+ke^{i\theta}\right)+ke^{i\theta}\right\}\geq\delta.$$

Up on choose the value of z on the positive real axis, where $0\leq z=r<1$, the above inequality reduces to

$$Re\left\{\frac{\left(\left((1-2\lambda)\left(1+ke^{i\theta}\right)+ke^{i\theta}+\delta\right)z^{-1}-\sum_{n=1}^{\infty}\left(n\left(n+\frac{\lambda}{3}(n-1)(n-2)(1+k)+k+\delta\right)\left(\frac{\beta}{n+\beta+1}\right)^\alpha\right)\right)}{z^{-1}-\sum_{n=1}^{\infty}\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n}\right\}\geq 0.$$

Since $Re\{-e^{i\theta}\}\geq-\left|e^{i\theta}\right|=-1$, the above inequality reduce to

$$\left\{\frac{\left((1-2\lambda)(1+k)+k+\delta)z^{-1}-\sum_{n=1}^{\infty}\left(n\left(n+\frac{\lambda}{3}(n-1)(n-2)(1+k)+k+\delta\right)\left(\frac{\beta}{n+\beta+1}\right)^\alpha\right)\right)}{z^{-1}-\sum_{n=1}^{\infty}\left(\frac{\beta}{n+\beta+1}\right)^\alpha a_nz^n}\right\}\geq 0$$

Letting $r\to 1^-$ and by the mean value Theorem, we have desired inequality (2.1).

Corollary (2.1): Let $f\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$. Then

$$a_n\leq\frac{2k(1-\lambda)-2\lambda+1+\delta}{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^\alpha},\qquad(n\geq 1)$$

3. Convex Linear Combination

In the following theorem, we prove the class $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$ is closed under convex linear combination.

Theorem (2): The class $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$is a closed under convex linear combination.

Proof: Let f_1 and f_2 be the arbitrary elements of $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$.
Then for every $t\ (0\le t\le 1)$, we show that$(1-t)f_1+tf_2\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$. Thus,we have

$$(1-t)f_1+tf_2=z^{-1}+\sum_{n=1}^{\infty}[(1-t)a_n+tb_n]z^n.$$

Hence,

$$\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}[(1-t)a_n+tb_n]$$
$$=(1-t)\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}a_n+t\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}b_n$$
$$\le(1-t)2k(1-\lambda)-2\lambda+1+\delta+t2k(1-\lambda)-2\lambda+1+\delta=2k(1-\lambda)-2\lambda+1+\delta.$$

This completes the proof.

4. Closure Theorem

We shall prove the closure theorem of the functions in the class $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$.

Theorem (4.3): Let the functions f_k defined by

$$f_k(z)=z^{-1}+\sum_{n=1}^{\infty}a_{n,k}\ ,\quad(a_{n,k}\ge 0,n\in N\ ,k=1,2,\dots,l)$$

be in the class $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$ for every $k-1,2,3,\dots,l$, then the function h defined by

$$h(z)=z^{-1}+\sum_{n=1}^{\infty}e_nz^n,\qquad(e_n\ge 0,n\in N)$$

also belong to the class $\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, where

$$e_n=\frac{1}{l}\sum_{n=1}^{\infty}a_{n,k}$$

.

Proof: Since $f_k\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$ it follows from Theorem (1) that

$\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\right.$
δβn+β+1αan,k≤2k1−λ−2λ+1+δ.

For every $k=1,2,3,\dots,l$. Hence

$$\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}e_n$$
$$=\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left(\frac{1}{l}\sum_{k=1}^{l}a_{n,k}\right)$$
$$=\frac{1}{l}\sum_{k=1}^{l}\left(\sum_{n=1}^{\infty}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\right)$$
$$\le\frac{1}{l}\sum_{k=1}^{l}2k(1-\lambda)-2\lambda+1+\delta=2k(1-\lambda)-2\lambda+1+\delta$$

Then $h\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$.

5. Partial Sums and Neighborhood Property

We introduce the partial sums and the same property has been found for other class in [8]

Theorem (5.6): Let $f\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$ be given by (1.1) and define the partial sums $S_1(z)$ and $S_k(z)$ as follows $S_1(z)=z^{-1}$ and

$$S_n(z)=z^{-1}-\sum_{n=1}^{k-1}a_n\,z^n,(k\in\mathbb{N}\setminus\{1\}).\qquad(5.1)$$

Also suppose that$\sum_{n=1}^{\infty}d_na_n\le$
1
dn=n1+kn+λ3n−1n−2+k+δβn+β+1α2k1−λ−2λ+1+δ.
(5.2)

Then, we have

$$Re\left\{\frac{f(z)}{S_k(z)}\right\}>1-\frac{1}{d_k}(z\in U,k\in\mathbb{N})\qquad(5.3)$$

and

$$Re\left\{\frac{S_k(z)}{f(z)}\right\}>\frac{d_k}{1+d_k}(z\in U,k\in\mathbb{N}).\qquad(5.4)$$

Each of the bounds in (5.3) and (5.4) is the best possible for $n\in\mathbb{N}$.

Proof: For the coefficients d_n given by (5.2), it is not difficult to verify that

$$d_{n+1}>d_n>1\ \ (n\in\mathbb{N}).$$

Therefore, by using hypothesis (5.2), we have

$$\sum_{n=1}^{k-1}a_n+d_k\sum_{n=k}^{\infty}a_n\le\sum_{n=1}^{\infty}d_na_n\le 1.\qquad(5.5)$$

By setting

$$g_1(z) = d_k \left[\frac{f(z)}{S_k(z)} - \left(1 - \frac{1}{C_k}\right) \right]$$
$$= 1 - \frac{d_k \sum_{n=1}^{\infty} a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} a_n z^{n+1}}, \qquad (5.6)$$

and applying (5.5), we find that

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq \frac{d_k \sum_{n=1}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n - d_k \sum_{k=n}^{\infty} a_n}$$
$$\leq 1 \quad (z \in U), \qquad (5.7)$$

which readily yields the assertion (5.2). If we take

$$f(z) = z^{-1} - \frac{z^k}{C_k}, \qquad (5.8)$$

then

$$\frac{f(z)}{S_k(z)} = 1 - \frac{z^k}{C_k} \longrightarrow 1 - \frac{1}{C_k} (z \longrightarrow 1^-),$$

which shows that the bound in (5.3) is the best possible for $n \in \mathbb{N}$.

Similarly, if we put

$$g_2(z) = (1 + d_k) \left[\frac{S_k(z)}{f(z)} - \frac{d_k}{1 + C_k} \right]$$
$$= 1 - \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n z^{n+1}}{1 - \sum_{n=1}^{\infty} a_k z^{k+1}}, \qquad (5.9)$$

and make use of (5.5), we have

$$\left|\frac{g_2(z) - 1}{g_2(z) + 1}\right|$$
$$\leq \frac{(1 + d_k) \sum_{n=k}^{\infty} a_n}{2 - 2\sum_{n=1}^{k-1} a_n + (1 + d_k) \sum_{n=k}^{\infty} a_n}, \qquad (5.10)$$

which leads us to the assertion (5.4). The bound in (5.4) is sharp for each $k \in \mathbb{N}$ with function given by (5.8). The proof of the theorem is complete.

Now, following the earlier works on neighborhoods of analytic functions by Goodman [5] and Ruscheweyh [9] investigated this concept for the element of several famous subclasses of analytic function and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [8] and Atshan [3] extended this concept for a certain subclass of meromorphically univalent and multivalent functions.

We begin by introducing here the (n, γ) −neighborhood of a function $f \in \Sigma$ of the form (1.1) by means of the definition below:

$$N_{n,\gamma}(f) = \left\{ g \in \Sigma : g(z) = z^{-1} - \sum_{n=1}^{\infty} b_n z^n \, and \sum_{n=1}^{\infty} n|a_n - b_n| \leq \gamma, 0 \leq \gamma < 1 \right\}. \qquad (5.11)$$

Particulary for the identity function $e(z) = z^{-1}$, we have

$$N_\delta(e) = \left\{ g \in \Sigma : g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n \, and \sum_{n=1}^{\infty} n|b_n| \leq \gamma, 0 \leq \gamma < 1 \right\}. \qquad (5.12)$$

Definition (5.1): A function $f \in \Sigma$ is said to be in the class $\mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$ if there exists a function $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$, such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \eta (z \in U, 0 \leq \eta < 1).$$

Theorem (5.1): If $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$ and

$$\eta = 1 - \frac{\gamma((1 + k)(k + \delta + 1))\left(\frac{\beta}{n + \beta + 1}\right)^\alpha}{((1 + k)(k + \delta + 1))\left(\frac{\beta}{n + \beta + 1}\right)^\alpha - (2k(1 - \lambda) - 2\lambda + (1 + \delta))}, \qquad (5.13)$$

then $N_{n,\gamma}(g) \subset \mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$.

Proof: Let $f \in N_{n,\gamma}(g)$. Then we find from (5.11) that

$$\sum_{k=1}^{\infty} k|a_k - b_k| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad (k \in \mathbb{N}). \qquad (5.14)$$

Since $g \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$, then by using Theorem (1), we get

$$\sum_{n=1}^{\infty} b_n \leq \frac{2k(1 - \lambda) - 2\lambda + (1 + \delta)}{(1 + k)(1 + k + \delta)\left(\frac{\beta}{n + \beta + 1}\right)^\alpha}, \qquad (5.15)$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n}$$
$$\leq \frac{\gamma((1 + k)(k + \delta + 1))\left(\frac{\beta}{n + \beta + 1}\right)^\alpha}{((1 + k)(k + \delta + 1))\left(\frac{\beta}{n + \beta + 1}\right)^\alpha - (2k(1 - \lambda) - 2\lambda + (1 + \delta))}.$$
$$= 1 - \eta$$

Hence, by Definition (5.1), $f \in \mathcal{A}^\eta(\delta, \lambda, k, \alpha, \beta)$ for η given by (5.13). This completes the proof.

6. The Radii of Starlikess and Convexity

In the following theorems, we obtain the radii of starlikeness and convexity for $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$.

Theorem (6.1): If $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, then f is univalent meromorphic starlike of order $\varphi(0 \le \varphi < 1)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_k \left\{ \frac{(1-\varphi)\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(n-\varphi+2)2k(1-\lambda)-2\lambda+1+\delta} \right\}^{\frac{1}{n+1}}.$$

The result is sharp for the function f given

$$f(z) = z^{-1} - \frac{2k(1-\lambda)-2\lambda+1+\delta}{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}} \qquad (6.1)$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le 1-\varphi \quad \text{for} |z| < r_1. \qquad (6.2)$$

But

$$\left|\frac{zf'(z)}{f(z)}+1\right| = \left|\frac{zf'(z)+f(z)}{f(z)}\right| \le \frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}}.$$

Thus, (6.2) will be satisfied if

$$\frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}a_n|z|^{n+1}} \le 1-\varphi,$$

or if

$$\sum_{n=1}^{\infty}\frac{(n-\varphi+2)}{1-\varphi}a_n|z|^{n+1} \le 1. \qquad (6.3)$$

Since $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+1+\delta}a_n \le 1.$$

Hence, (6.3) will be true if

$$\frac{(n-\varphi+2)}{1-\varphi}|z|^{k+1} \le \frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+1+\delta},$$

or equivalently

$$|z| \le \left\{ \frac{(1-\varphi)\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(n-\varphi+2)2k(1-\lambda)-2\lambda+1+\delta} \right\}^{\frac{1}{n+1}} \quad (n \ge 1),$$

which follows the result.

Theorem (6.2): If $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, then f is univalent meromorphic convex of order $\varphi(0 \le \varphi < 1)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{(1-\varphi)\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(k-\varphi+1)2k(1-\lambda)-2\lambda+1+\delta} \right\}^{\frac{1}{k+1}}.$$

The result is sharp for the function f given by (3.9).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \le 1-\varphi \quad \text{for} |z| < r_2. \qquad (6.4)$$

But

$$\left|\frac{zf''(z)}{f'(z)}+2\right| = \left|\frac{zf''(z)+2f'(z)}{f'(z)}\right| \le \frac{\sum_{n=1}^{\infty}n^2a_k|z|^{n+1}}{1-\sum_{n=1}^{\infty}na_n|z|^{n+1}}.$$

Thus, (6.4) will be satisfied if

$$\frac{\sum_{n=1}^{\infty}n^2a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}na_n|z|^{n+1}} \le 1-\varphi,$$

or if

$$\sum_{n=1}^{\infty} \frac{n(n-\varphi+1)}{1-\varphi} a_n |z|^{n+1} \leq 1. \qquad (6.5)$$

Since $f \in \Sigma(\alpha, \gamma, \lambda)$, we have

$$\sum_{n=1}^{\infty} \frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+1+\delta} a_n \leq 1.$$

Hence, (6.5)will be true if

$$\frac{k(k-\varphi+1)}{1-\varphi}|z|^{k+1} \leq \frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+1+\delta},$$

or equivalently

$$|z| \leq \left\{\frac{(1-\varphi)\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(k-\varphi+1)2k(1-\lambda)-2\lambda+1+\delta}\right\}^{\frac{1}{k+1}} \quad (n \geq 1),$$

which follows the result.

7. Integral Transformation

In the following theorems, we obtain integral transformations in the class $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$.

Theorem (7.1): Let the function f given by (1.1) be in the class $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$. Then the integral operator

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt, \qquad (0 < t \leq 1, 0 < c < \infty)$$

is in the class $\mathcal{A}(\sigma, \lambda, k, \alpha, \beta)$, where.

$$\sigma \leq \frac{(c+n+1)(2k(1-\lambda)-2\lambda+1)y - cw(n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k)}{(w-(c+n+1))y}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} - \frac{2k(1-\lambda)-2\lambda+\delta+1}{(2k+\delta+1)\left(\frac{\beta}{2+\beta}\right)^{\alpha}} z$$

Proof : Let

$$f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} a_n z^n$$

in the class $\mathcal{A}(\delta, \lambda, k, \alpha, \beta)$. Then

$$F_c(f(z)) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt$$

$$= \frac{c}{z^{c+1}} \int_0^z \left[t^{c-1} - \sum_{n=1}^{\infty} a_n t^{n+1}\right]$$

$$= \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{c}{c+n+1}\right) a_n \qquad (7.2)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\sigma\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(c+n+1)(2k(1-\lambda)-2\lambda+\sigma+1)} a_n \leq 1. \qquad (7.3)$$

Since $f \in \mathcal{A}(\delta, \lambda, k, \alpha, \beta)$, we have

$$\sum_{n=1}^{\infty} \frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+\delta+1} a_n \leq 1. \qquad (7.4)$$

Note that (7.1) it satisfies if

$$\frac{c\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\sigma\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{(c+n+1)(2k(1-\lambda)-2\lambda+\sigma+1)} \le \frac{\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}{2k(1-\lambda)-2\lambda+\delta+1}.$$

Rewriting the inequality, we have

$$c(2k(1-\lambda)-2\lambda+\delta+1)\left(n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\sigma\right)$$
$$\le (c+n+1)(2k(1-\lambda)-2\lambda+\sigma+1)\left(n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right).$$

Solving for σ, we have

$$\sigma \le \frac{(c+n+1)(2k(1-\lambda)-2\lambda+1)y-cw(n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k)}{(w-(c+n+1))y}$$
$$= F(n),$$

where

$w = 2k(1-\lambda)-2\lambda+\delta+1$ and $y = n\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)(1+k)+k+\delta)$

$$F(n) \ge F(1).$$

Using this, the results follows.

Theorem (7.2) : If $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, then the integral operator

$$F_c(f(z)) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt, \quad (0 < t \le 1, 0 < c < \infty)$$

is in the class $f \in \mathcal{A}\left(\frac{1+\delta c}{2+c},\lambda,k,\alpha,\beta\right)$,

The result is sharp for

$$f_n(z) = z^{-1} - \frac{2k(1-\lambda)-2\lambda+1(1+\left(\frac{1+\delta c}{2+c}\right))}{n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\left(\frac{1+\delta c}{2+c}\right)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}z^n.$$

Proof: By definition of F_c, we obtain

$$F_c(f(z)) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt = \frac{1}{z} - \sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n z^n.$$

By Theorem (2.1), it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\left(\frac{1+\delta c}{2+c}\right)\right]}{(c+n+1)\left(2k(1-\lambda)-2\lambda+1+\frac{1+\delta c}{2+c}\right)}a_n \le 1. \quad (7.5)$$

Since, if $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$, then (7.5) satisfied if

$$\frac{c}{(c+n+1)\left(2k(1-\lambda)-2\lambda+1+\frac{1+\delta c}{2+c}\right)} \le \frac{1}{2k(1-\lambda)-2\lambda+1+\delta}.$$

Or equivalent, when

$$\phi(n,\lambda,k,\delta,c) = \frac{c(2k(1-\lambda)-2\lambda+\delta+1)}{(c+n+1)\left(2k(1-\lambda)-2\lambda+\left(1+\frac{1+\delta c}{2+c}\right)\right)} \le 1.$$

Since $\phi(n,\lambda,k,\delta,c)$ is a decreasing function of n $(n \ge 1)$. Then the proof is complete. The result sharp for the function

$$f_n(z) = z^{-1} - \frac{2k(1-\lambda)-2\lambda+1(1+\left(\frac{1+\delta c}{2+c}\right))}{n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\left(\frac{1+\delta c}{2+c}\right)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}}z^n$$

Theorem (7.3) : Let $f \in \mathcal{A}(\delta,\lambda,k,\alpha,\beta)$. Then the function F defined by

$$F_c(f(z)) = \frac{c}{z^{c+1}}\int_0^z t^c f(t)dt = \frac{1}{z} - \sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n z^n, \quad 0 < c < \infty \qquad (7.6)$$

is meromorphically starlike in the disk $|z| < R_1$, where

$$R_1 = inf_n\left\{\frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{c(n+2)(2k(1-\lambda)-2\lambda+\delta+1)}\right\}^{\frac{1}{n+1}}. \qquad (7.7)$$

Proof : We show that

$$\left|\frac{zF'_c(z)}{F_c(z)}+1\right|\leq 1 \ \text{ in } |z|<R_1. \tag{7.8}$$

R_1 is given by (7.7). In view of (7.6) we have

$$\left|\frac{zF'_c(z)+F_c(z)}{F_c(z)}\right| = \left|\frac{-\sum_{n=1}^{\infty}\left(\frac{c}{c+n+1}\right)(n+1)a_n z^{n+1}}{1-\sum_{n=1}^{\infty}\frac{c}{(c+n+1)}a_n z^{n+1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}\left(\frac{c(n+1)}{c+n+1}\right)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}\frac{c}{(c+n+1)}a_n|z|^{n+1}}.$$

Thus (7.8) will be satisfied if

$$\sum_{n=`}^{\infty}\frac{c(n+2)}{(c+n+1)}a_n|z|^{n+1}\leq 1. \tag{7.9}$$

Hence (7.9) will be true if

$$|z|^{n+1}\leq\left\{\frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{c(n+2)(2k(1-\lambda)-2\lambda+\delta+1)}\right\}.$$

Therefore

$$|z|\leq\left\{\frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{c(n+2)(2k(1-\lambda)-2\lambda+\delta+1)}\right\}^{\frac{1}{n+1}}.$$

For $n\geq 1, n\in N$, The result follows by setting $|z|=R_1$

Theorem (7.4) : Let $f\in\mathcal{A}(\delta,\lambda,k,\alpha,\beta)$. Then the function F defined by

$$F_c(f(z))=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt=\frac{1}{z}-\sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n z^n,\quad 0<c<\infty \tag{7.10}$$

is convex in the disk $|z|<R_2$, where

$$R_2=inf_n\left\{\frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{cn^2(2k(1-\lambda)-2\lambda+\delta+1)}\right\}^{\frac{1}{n+1}} \tag{7.11}$$

Proof : We show that

$$\left|\frac{zF'_c(z)}{F_c(z)}+2\right|\leq 1 \ \text{ in } |z|<R_2, \tag{7.12}$$

R_2 is given by (7.7). In view of (7.6) we have

$$\left|\frac{zF'_c(z)+2F_c(z)}{F_c(z)}\right| = \left|\frac{-\sum_{n=1}^{\infty}\left(\frac{cn^2}{c+n+1}\right)a_n z^{n+1}}{1-\sum_{n=1}^{\infty}\frac{cn}{(c+n+1)}a_n z^{n+1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}\left(\frac{cn^2}{c+n+1}\right)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}\frac{cn}{(c+n+1)}a_n|z|^{n+1}}.$$

Thus (7.12) will be satisfied if

$$\sum_{n=`}^{\infty}\frac{cn^2}{(c+n+1)}a_n|z|^{n+1}\leq 1. \tag{7.13}$$

Hence (7.13) will be true if

$$|z|^{n+1}\leq\left\{\frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{cn^2(2k(1-\lambda)-2\lambda+\delta+1)}\right\}.$$

Therefore

$$|z| \leq \left\{ \frac{(c+n+1)\left(\frac{\beta}{n+\beta+1}\right)^{\alpha}\left[n(1+k)\left(n+\frac{\lambda}{3}(n-1)(n-2)\right)+k+\delta\right]}{cn^2(2k(1-\lambda)-2\lambda+\delta+1)} \right\}^{\frac{1}{n+1}}.$$

For $n \geq 1, n \in N,$ The result follows by setting $|z| = R_2$

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