Some Certain Properties of a New Class of p-valent Functions Defined by Komatu Operator

Waggas Galib Atshan[1], Safa Ehab mohammed[2]

Department of Mathematics, College of Computer Science and Mathematics, University of Al-Qadisiya, Diwaniya – Iraq

Abstract: ***In the present paper ,we introduced a new class $W(\gamma,p,c,A,B,\delta,v)$ of p-valent functions in the unit disk U={z∈ ℂ:IzI<1} defined by Komatu operator .We obtain some certain properties ,like, coefficient inequality ,distortion theorem ,radii of starlikeness and convexity ,Hadamard product properties ,convex linear combinations and integral mean inequalities for the fractional integral.***

AMS Subject Classification: 30C45

Keywords: p-valent function, Komatu operator, distortion theorem, radius of starlikeness, Hadamard product, integral mean.

1. Introduction

Let M_p denote the class of the functions of the form:

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\ ,(a_{n+p}\geq 0\ p\in\mathbb{N}=\{1,2,\dots\})\ ,(1.1)$$

which are analytic and p-valent in the unit disk U.

A function $f\in M_p$ is said to be in the class $S^*(\alpha)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ ,(0\leq\alpha<1\ ;z\in U).\ (1.2)$$

The elements of this class are called starlike functions of order α.

A function $f\in M_p$ is said to be in the class $C(\alpha)$ if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\ ,(0\leq\alpha<p;z\in U).\ (1.3)$$

The elements of this class are called convex functions of order.

We shall use the Komatu operator[4] $K_{C,p}^{\delta}$ of the function f given by (1.1) defined by

$$K_{C,p}^{\delta}f(z)=\frac{(c+p)^{\delta}}{\Gamma(\delta)z^{C}}\frac{d}{dz}\int_0^z t^{C-1}\left(\log\frac{z}{t}\right)^{\delta-1}f(t)dt\ ,$$

$$=z^p-\sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)^{\delta}a_{n+p}z^{n+p}\quad(c>-p,\delta>0)\ (1.4)$$

Definition(1.1): Let $f\in M_p$ given by(1.1).Then the class $W(\gamma,p,c,A,B,\delta,v)$ is defined by

$$W(\gamma,p,c,A,B,\delta,v)$$

$$=\left\{f\in M_p:\left|\frac{p\left(\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}-1\right)}{\gamma\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}+(A+B+\gamma)}\right|<v\ ,0\leq\gamma<1,0<A\leq 1\ ,0\leq B<1\ and\ 0<v<1\right\}.\ (1.5)$$

The p-valent functions studied by several authors for another classes ,like,[1],[2],[3],[6], and [7].

2. Coefficient Inequality

The following theorem gives a sufficient and necessary condition for a function f to be in the class $W(\gamma,p,c,A,B,\delta,v)$.

Theorem(2.1): Let $f\in M_p$.Then $f\in W(\gamma,p,c,A,B,\delta,v)$ if and only if

$$\sum_{n=1}^{\infty}\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}a_{n+p}\leq v(A+B+2\gamma),(2.1)$$

where $0\leq\gamma<1\ ,0<A\leq 1\ ,0\leq B<1,0<v<1\ and\ p\in\mathbb{N}$.

proof : Suppose that the inequality (2.1) holds true and $|z|=1$. Then , we have

$$\left|p\left(\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}-1\right)\right|-v\left|\gamma\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}+(A+B+\gamma)\right|$$

$$= \left| \sum_{n=1}^{\infty} \left(\frac{pn}{p+c+n}\right)\left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p} z^{n+p} \right|$$
$$-v\left|(A+B+2\gamma)z^{p} - \sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\gamma\left(\frac{c+p}{c+p+n}\right)+(A+B+\gamma)\right)a_{n+p}z^{n+p}\right|$$
$$\leq \sum_{n=p+1}^{\infty}\left(\frac{pn}{p+c+n}\right)\left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p}|z|^{n} - v(A+B+2\gamma)|z|^{p}$$
$$+\sum_{n=1}^{\infty} v\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\gamma\left(\frac{c+p}{c+p+n}\right)+(A+B+\gamma)\right)a_{n+p}|z|^{n+p}$$
$$= \sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\left(\frac{pn}{p+c+n}\right)+v\gamma\left(\frac{c+p}{c+p+n}\right)+v(A+B+\gamma)^{\delta}\right)a_{n+p}|z|^{n+P}$$
$$- v(A+B+2\gamma)|z|^{p}$$
$$= \sum_{n=1}^{\infty}\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p} - v(A+B+2\gamma) \leq 0$$

by hypothesis .

Hence , by maximum modulus principle , $f \in W(\gamma,p,c,A,B,\delta,v)$.

Conversely , suppose that $f \in W(\gamma,p,c,A,B,\delta,v)$. Then from (1.5) , we have

$$\left|\frac{p\left(\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}-1\right)}{\gamma\frac{K_{C,p}^{\delta+1}f(z)}{K_{C,p}^{\delta}f(z)}+(A+B+\gamma)}\right|$$
$$= \left|\frac{\sum_{n=1}^{\infty}\left(\frac{pn}{p+c+n}\right)\left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p}z^{n+p}}{(A+B+2\gamma)z^{p}-\sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\gamma\left(\frac{c+p}{c+p+n}\right)+(A+B+\gamma)\right)a_{n+p}z^{n+p}}\right|$$
$$< v \,.$$

Since $Re(z) \leq |z|$ for all $(z \in U)$, we get

$$Re\left\{\frac{\sum_{n=1}^{\infty}\left(\frac{pn}{p+c+n}\right)\left(\frac{c+p}{c+p+n}\right)^{\delta} a_{n+p}z^{n+p}}{(A+B+2\gamma)z^{p}-\sum_{n=1}^{\infty}\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\gamma\left(\frac{c+p}{c+p+n}\right)+A+B+\gamma\right)a_{n+p}z^{n+p}}\right\} < v\,.$$

We choose the value of z on the real axis and Letting $z \to 1^{-}$, through real values , we obtain the inequality (2.1) .

Corollary(2.1): Let $f \in W(\gamma,p,c,A,B,\delta,v)$. Then

$$a_n \leq \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}} \,. \quad (2.2)$$

The equality in (2.2)is attained for the function f given by the sharp for the function

$$f(z) = z^{p} - \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}} z^{n+p} \quad (2.3)$$

3. Distortion Theorem

In the following theorem , we obtain the distortion bounds for the function f in the class $W(\gamma,p,c,A,B,\delta,v)$.

Theorem (3.1): Let the function $f \in W(\gamma,p,c,A,B,\delta,v)$.Then

$$|z|^p - \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}|z|^{p+1} \le |f(z)|$$
$$\le |z|^p + \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}|z|^{p+1}.$$

(3.1)

The result is sharp for the function f given by

$$f(z) = z^p - \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}z^{p+1}$$

Proof: We have
$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$, then
$|f(z)| \le |z|^p + \sum_{n=1}^{\infty} a_{n+p}|z|^{p+n}$ and since $f \in W(\gamma,p,c,A,B,\delta,v)$,
then by Theorem(2.1) , we have
$\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^\delta \sum_{n=1}^{\infty} a_{n+p} \le$

$$\sum_{n=1}^{\infty}\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta a_{n+p} \le v(A+B+2\gamma).$$

Hence

$$\sum_{n=1}^{\infty} a_{n+p} \le \frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}. \quad (3.2)$$

Thus $|f(z)| \le |z|^p + |z|^{p+n}\sum_{n=1}^{\infty} a_{n+p}$

$$\le |z|^p + |z|^{p+n}\frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}. \quad (3.3)$$

Similarly , we have

$$|f(z)| \ge |z|^p - \sum_{n=1}^{\infty} a_{n+P}|z|^{p+n}.$$
$$\ge |z|^p - |z|^{p+n}\sum_{n=1}^{\infty} a_{n+p}.$$

$$\ge |z|^p - |z|^{p+n}\frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^\delta}. \quad (3.4)$$

From Combining (3.3) and (3.4), we get (3.1).

4. Radii of starlikeness and convexity

In the following theorems, we obtain the radii of starlikeness and convexity of the class $W(\gamma,p,c,A,B,\delta,v)$.

Theorem (4.1): Let $f \in W(\gamma,p,c,A,B,\delta,v)$Then f is multivalent starlike of order $\beta, (0 \le \beta < p)$ in the disk $|z| < r = r_1(\gamma,p,c,A,B,\delta,\beta,v)$, where

$$r_1(\gamma,p,c,A,B,\delta,\beta,v) = \inf_n \left\{\frac{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](p-\beta)\left(\frac{c+p}{c+p+n}\right)^\delta}{(n+p-\beta)(v(A+B+2\gamma)}\right\}^{\frac{1}{n}},$$

$$n = 1,2,\ldots . (4.1)$$

The result is sharp for the function f given by (2.3).

Proof: It is sufficient to show that

$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \beta$, for $|z| < r_1$. (4.2)

But

$$\left|\frac{zf'(z)}{f(z)}-p\right|=\left|\frac{zf'(z)-pf(z)}{f(z)}\right|=\left|\frac{-\sum_{n=1}^{\infty} na_{n+p}z^{n+p}}{z^p-\sum_{n=1}^{\infty} a_{n+p}z^{n+p}}\right|\leq\frac{\sum_{n=1}^{\infty} na_{n+p}a_n|z|^n}{1-\sum_{n=1}^{\infty} a_{n+p}|z|^n}.$$

Thus , (4.2) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} na_{n+p}a_n|z|^n}{1-\sum_{n=1}^{\infty} a_{n+p}|z|^n}\leq p-\beta,$$

or if

$\sum_{n=1}^{\infty}\left(\frac{n+p-\beta}{p-\beta}\right)a_{n+p}|z|^n\leq 1.$ (4.3)

Since $f\in \mathrm{W}(\gamma,p,c,A,B,\delta,v)$,we have

$$\sum_{n=1}^{\infty}\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^{\delta}}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)}a_{n+p}\leq 1\,.$$

Hence , (4.3) will be true if

$$\frac{n+p-\beta}{p-\beta}|z|^n\leq\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](\frac{c+p}{c+p+n})^{\delta}}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)},$$

or equivalently

$$|z|\leq\left\{\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right](p-\beta)\left(\frac{c+p}{c+p+n}\right)^{\delta}}{(n+p-\beta)(\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)}\right\}^{\frac{1}{n}},$$

$n\geq 1$.

which follows the result.

Theorem (4.2): Let $f\in \mathrm{W}(\gamma,p,c,A,B,\delta,v)$Then f is multivalent convex of order β , $(0\leq\beta<p)$ in $|z|<r=r_2(\gamma,p,c,A,B,\delta,\beta,v)$,where

$$r_2(\gamma,p,c,A,B,\beta,\delta,v)=\inf_n\left\{\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\mathrm{p}(p-\beta)\left(\frac{c+p}{c+p+n}\right)^{\delta}}{(n+p)(n+p-\beta)\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)}\right\}^{1/n},n\geq 1$$

The result is sharp for the function f given by (2.3).

Proof: It is sufficient to show that

$\left|\frac{zf''(z)}{f'(z)}+1-p\right|\leq p-\beta$, for $|z|<r_2$. (4.4)

But

$$\left|\frac{zf''(z)}{f'(z)}+1-p\right|=\left|\frac{zf''(z)+(1-p)f'(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=1}^{\infty} n(n+p)a_{n+p}z^{n+p}}{pz^p-\sum_{n=1}^{\infty}(n+p)a_{n+p}z^{n+p}}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z|^n}$$

Thus , (4.4) will be satisfied if

$$\frac{\sum_{n=1}^{\infty} n(n+p)a_{n+p}|z|^n}{p-\sum_{n=1}^{\infty}(n+p)a_{n+p}|z|^n} \leq p-\beta,$$

or if

$$\sum_{n=1}^{\infty}\left(\frac{(n+p)(n+p-)\beta}{p(p-\beta)}\right)a_{n+p}|z|^n \leq 1. \ (4.5)$$

Since $f \in W(\gamma,p,c,A,B,\delta,v)$,we have

$$\sum_{n=1}^{\infty}\frac{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+B+2\gamma)}a_{n+p} \leq 1 .$$

Hence , (4.5) will be true if

$$\frac{(n+p)(n+p-)\beta}{p(p-\beta)}|z|^n \leq \frac{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+B+2\gamma)}.$$

or equivalently

$$|z| \leq \left\{\frac{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]p(p-\beta)\left(\frac{c+p}{c+p+n}\right)^{\delta}}{(n+p)(n+p-\beta)v(A+B+2\gamma)}\right\}^{\frac{1}{n}},$$

$$n \geq 1 .$$

Which follows the result.

5. Convex linear Combination

Theorem (5.1): The class $W(\gamma,p,c,A,B,\delta,v)$is closed under convex linear combinations.

Proof: Let f and g be the arbitrary elements of $W(\gamma,p,c,A,B,\delta,v)$Then for every t $(0<t<1)$, we show that $(1-t)f(z)+tg(z) \in W(\gamma,p,c,A,B,\delta,v)$. Thus , we have

$$(1-t)f(z)+tg(z) = z^p - \sum_{n=1}^{\infty}[(1-t)\,a_{n+p} + t\,b_{n+p}]z^{p+n}.$$

Therefore

$$\sum_{n=1}^{\infty}\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\ [(1-t)a_{n+p} + t\,b_{n+p}]$$

$$= (1-t)\sum_{n=1}^{\infty}\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\ a_{n+P}$$

$$+\ t\sum_{n=1}^{\infty}\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\ b_{n+p}$$

$$\leq (1+t)v(A+B+2\gamma)+tv(A+B+2\gamma) = v(A+B+2\gamma)$$

This completes the proof.

6. Hadamard Product Properties:

Theorem (6.1): Let the functions f_j $(j=1,2)$defined by

$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{n+p,j}\,z^{n+p}$, $(a_{n+p,j} \geq 0, p\in \mathbb{N}, j=1,2)$ (6.1)

be in the class $W(\gamma,p,c,A,B,\delta,v)$.Then $f_1 * f_2 \in W(\gamma,p,c,A,\sigma,\delta,v)$,where

$$\sigma \leq \frac{(A+2\gamma)\left(\frac{c+p}{c+p+n}\right)^{\delta} - v(c+p+n)(v\gamma(c+p)+pn+v(A+B+\gamma)(A+\gamma))}{v^2(c+p+n)^2 - \left(\frac{c+p}{c+p+n}\right)^{\delta}}.$$

Proof: We have to find the largest σ such that

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{v(\gamma(c+p)+(A+\sigma+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+\sigma+2\gamma)}\right]a_{n+p,1}a_{n+p,2} \leq 1.$$

Since $f_j \in W(\gamma,p,c,A,B,\delta,v)(j=1,2)$,we get

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+B+2\gamma)}\right]a_{n+p,j} \leq 1. (6.2)$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+B+2\gamma)}\right]\sqrt{a_{n+p,1}a_{n+p,2}} \leq 1. (6.3)$$

We want only to show that

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{v(\gamma(c+p)+(A+\sigma+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+\sigma+2\gamma)}\right]a_{n+p,1}a_{n+p,2}$$
$$\leq \sum_{n=1}^{\infty}\left[\frac{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{v(A+B+2\gamma)}\right]\sqrt{a_{n+p,1}a_{n+p,2}}.$$

This equivalently to

$$\sqrt{a_{n+p,1}a_{n+p,2}} \leq \frac{(A+\sigma+2\gamma)[v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn]}{(A+B+2\gamma)[v(\gamma(c+p)+(A+\sigma+\gamma)(c+p+n))+pn]}.$$

From (6.3), we get

$$\sqrt{a_{n+p,1}a_{n+p,2}} \leq \frac{v(A+B+2\gamma)}{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}.$$

Thus, it is sufficient to show that

$$\frac{v(A+B+2\gamma)}{\left[\frac{v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}} \leq \frac{(A+\sigma+2\gamma)[v\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn]}{(A+B+2\gamma)[v(\gamma(c+p)+(A+\sigma+\gamma)(c+p+n))+pn]},$$

which implies to

$$\sigma \leq \frac{(A+2\gamma)\left(\frac{c+p}{c+p+n}\right)^{\delta} - \mathrm{v}(c+p+n)\big(\mathrm{v}\gamma(c+p)+pn+\mathrm{v}(A+B+\gamma)(A+\gamma)\big)}{\mathrm{v}^2(c+p+n)^2-\left(\frac{c+p}{c+p+n}\right)^{\delta}}$$

Theorem (6.2): Let the functions $f_j\,(\mathrm{j}=1,2)$ defined by (6.1) be in the class $\mathrm{W}(\gamma,p,c,A,B,\delta,v)$. Then, the function h defined by

$$h(z)=z^p-\sum_{n=1}^{\infty}\left(a_{n+p,1}^2+a_{n+p,2}^2\right)z^{n+p} \quad (6.4)$$

belongs to the class $\mathrm{W}(\gamma,p,c,A,\varepsilon,\delta,v)$, where

$$\varepsilon \leq \frac{(A+2\gamma)\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\left(\mathrm{v}\big(\gamma(c+p)+(A+B+\gamma)(c+p+n)\big)+pn\right)^2-2\mathrm{v}(A+B+2\gamma)^2(\mathrm{v}(c+p+n)(A+\gamma)+\mathrm{v}(A+B+\gamma)+pn)\right)}{2\mathrm{v}^2(A+B+2\gamma)^2(c+p+n)-\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(\mathrm{v}\big(\gamma(c+p)+(A+B+\gamma)(c+p+n)\big)+pn\right)^2}.$$

Proof: We must find the largest ε such that

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+\varepsilon+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+\varepsilon+2\gamma)}\right]\left(a_{n+p,1}^2+a_{n+p,2}^2\right)\leq 1.$$

Since $f_j \in \mathrm{W}(\gamma,p,c,A,B,\delta,v)\,(\mathrm{j}=1,2)$, we get

$$\sum_{n=1}^{\infty}\left(\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+B+2\gamma)}\right)^2 a_{n+p,1}^2$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+B+2\gamma)}a_{n+p,1}\right)^2 \leq 1, \quad (6.5)$$

and

$$\sum_{n=1}^{\infty}\left(\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+B+2\gamma)}\right)^2 a_{n+p,2}^2$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+B+2\gamma)}a_{n+p,2}\right)^2 \leq 1. \quad (6.6)$$

Combining the inequalities(6.5)and(6.6) , gives

$$\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+B+2\gamma)}\right)^2\left(a_{n+p,1}^2+a_{n+p,2}^2\right)\leq 1\,. \quad (6.7)$$

But $h \in \mathrm{W}(\gamma,p,c,A,\varepsilon,\delta,v)$, if and only if

$$\sum_{n=1}^{\infty}\left[\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+\varepsilon+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+\varepsilon+2\gamma)}\right]\left(a_{n+p,1}^2+a_{n+p,2}^2\right)\leq 1. \quad (6.8)$$

The inequality(6.8) will be satisfied if

$$\frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+\varepsilon+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}{\mathrm{v}(A+\varepsilon+2\gamma)} \leq \frac{\left[\frac{\left(\mathrm{v}\big(\gamma(c+p)+(A+B+\gamma)(c+p+n)\big)+pn\right)^2}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{2\delta}}{2\mathrm{v}^2(A+B+2\gamma)^2},$$

(n=1,2,…), so that

$$\varepsilon \le \frac{(A+2\gamma)\left(\frac{c+p}{c+p+n}\right)^{\delta}\left((v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn)^2-2v(A+B+2\gamma)^2(v(c+p+n)(A+\gamma)+v(A+B+\gamma)+pn)\right.}{2v^2(A+B+2\gamma)^2(c+p+n)-\left(\frac{c+p}{c+p+n}\right)^{\delta}\left(v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn\right)^2}$$

7. Integral Mean Inequalities for the Fractional Integral

Definition(7.1)[8]: the fractional integral of order $\lambda(\lambda>0)$is defined for a function f by:

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt\,,(7.1)$$

Where f is an analytic function in a simply-connected region of the z-plane containing the region ,and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring log(z-1) to be real ,when Re(z-1)>0. In 1925, Littlewoods [5] proved the following subordination theorem:

Theorem(7.1) [5]: If f and g are analytic in U with $g\prec f$, then for ▯ >0
and $z=re^{i\vartheta}$ $(0<r<1)$

$$\int_0^{2\pi}|f(z)|^{▯}d\vartheta\le\int_0^{2\pi}|g(z)|^{▯}d\vartheta\,.(7.2)$$

Theorem(7.2): Let $f\in W(\gamma,p,c,A,\varepsilon,\delta,v)$ and suppose that $f_{n+p}$is defined by

$$f_{n+p}(z)=z^p-\frac{v(A+B+2\gamma)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}}z^{n+p}\,,$$

$(n\ge1).(7.3)$
Also let

$$\sum_{m=1}^{\infty}(m+p-\eta)_{\eta+1}a_{m+P}\le\frac{v(A+B+2\gamma)\Gamma(n+1)\Gamma(p+\lambda+\eta+2)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)\Gamma(p-\eta+1)}$$

(7.4)

For $0\le\eta\le$ m+p , λ>0, where $(m+p-\eta)_{\eta+1}$denote the Pochhammer symbol defined by
$(m+p-\eta)_{\eta+1}=(m+p-\eta)(m+p-\eta+1)\ldots$m+p. (7.5)
If there exists an analytic function w defined by

$$(w(z))^n=\frac{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)}{v(A+B+2\gamma)\Gamma(n+1)}\times$$

$\times\sum_{m=1}^{\infty}(m+p-\eta)_{\eta+1}H(m+p)a_{m+p}\,z^m$, (7.6)
where $m+p\ge\eta$ and
$H(m+p)=\frac{\Gamma(m+p-\eta)}{\Gamma(m+p+\eta+\lambda+1)}$,(λ> 0, m$\ge$1) (7.7)
Then , for z= $re^{i\vartheta}$ and $0<r<1$

$$\int_0^{2\pi}\left|D_z^{-\lambda-\eta}f(z)\right|^{▯}d\vartheta\le\int_0^{2\pi}\left|D_z^{-\lambda-\eta}f_n(z)\right|^{▯}d\vartheta\,.(\lambda>0,▯\ge1)\ (7.2)$$

Proof: Let

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{m+p}z^{m+p}.$$

For $\eta\ge0$ and definition (7.1) , we get

$$D_z^{-\lambda-\eta}f(z)=\frac{\Gamma(1)}{\Gamma(1+\lambda+\eta)}\left(1-\sum_{m=1}^{\infty}\frac{\Gamma(m+1)\Gamma(1+\lambda+\eta)}{\Gamma(1)\Gamma(m+\eta+\lambda+1)}a_{m+p}z^{m+p}\right)$$

$$=\frac{z^{P+\lambda-\eta}}{\Gamma(1+\lambda+\eta)}(1-\sum_{m=1}^{\infty}\Gamma(1+\lambda+\eta)(m-\eta)_{\eta+1}H(m)a_{m+p}\,z^m),$$

where

$H(m)=\frac{\Gamma(m-\eta)}{\Gamma(m+\eta+\lambda+1)}$,(λ> 0, m$\ge$1).
Since H is a decreasing function of m , we have
$0<H(m)\le H(1)=\frac{\Gamma(1-\eta)}{\Gamma(\eta+\lambda+1)}$.
Similarly , from (7.3) and definition(7.1) ,we get

$$D_z^{-\lambda-\eta}f(z)=\frac{\Gamma(1)z^{P+\lambda-\eta}}{\Gamma(1+\lambda+\eta)}\left(1-\frac{v(A+B+2\gamma)\Gamma(n+1)\Gamma(1+\lambda+\eta)}{\left[\frac{v(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(1)\Gamma(n+\eta+\lambda+1)}z^n\right),$$

for ? >0 and $z = re^{i\vartheta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{m=1}^{\infty} \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1)} (m-\eta)_{\eta+1} H(m) a_{m+P} z^m \right|^{?} d\vartheta$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)\Gamma(1+\lambda+\eta)}{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(1)\Gamma(n+\eta+\lambda+1)} z^n \right|^{?} d\vartheta .$$

by applying Littlewood's Subordination Theorem , it would suffice to show that

$$1 - \sum_{m=1}^{\infty} \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1)} (m-\eta)_{\eta+1} H(m) a_{m+p} z^m \prec$$

$$1 - \frac{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)\Gamma(1+\lambda+\eta)}{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(1)\Gamma(n+\eta+\lambda+1)} z^n .$$

By setting

$$1 - \sum_{m=1}^{\infty} \frac{\Gamma(1+\lambda+\eta)}{\Gamma(1)} (m-\eta)_{\eta+1} H(m) a_{m+p} z^m$$

$$= 1 - \frac{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)\Gamma(1+\lambda+\eta)}{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(1)\Gamma(n+\eta+\lambda+1)} (w(z))^n .$$

We find that

$$(w(z))^n = \frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)} \times$$

$$\times \sum_{m=1}^{\infty} (m+\eta)_{\eta+1} H(m) a_{m+p} z^m ,$$

which readily yields w(0). For such a function w ,we obtain

$$|w(z)]|^n \leq \frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)} \times$$

$$\times \sum_{m=1}^{\infty} (m-\eta)_{\eta+1} H(m) a_{m+p} z^m$$

$$\leq \frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)} \times$$

$$\times H(1) |z| \sum_{m=1}^{\infty} (m+\eta)_{\eta+1} a_{m+p}$$

$$= |z| \frac{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\eta+\lambda+1)\Gamma(1-\eta)}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)\Gamma(2+\eta+\lambda)} \times$$

$$\times \sum_{m=1}^{\infty} (m+\eta)_{\eta+1} a_{m+p} \leq |z| < 1.$$

This completes the proof of the theorem.

By taking η=0 in the Theorem (7.2) , we have the following corollary:-

Corollary(7.1): Let $f \in W(\gamma, p, c, A, \varepsilon, \delta, v)$ and suppose that f_n is defined by (7.3). Also let

$$\sum_{m=1}^{\infty} (m+p) a_{m+P} \leq \frac{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)\Gamma(2+\lambda)}{\left[\frac{\mathrm{v}(\gamma(c+p)+(A+B+\gamma)(c+p+n))+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\lambda+1)},$$

$$(\mathrm{n} \geq 1).$$

If there exists an analytic function , w is defined by

$$(w(z))^n = \frac{\left[\frac{\mathrm{v}\left(\gamma(c+p)+(A+B+\gamma)(c+p+n)\right)+pn}{(c+p+n)}\right]\left(\frac{c+p}{c+p+n}\right)^{\delta}\Gamma(n+\lambda+1)}{\mathrm{v}(\mathrm{A}+\mathrm{B}+2\gamma)\Gamma(n+1)}\times$$

$$\times\sum_{m=1}^{\infty}(m+p)_{\eta+1}H(m+p)a_{m+P}\,z^m,$$

where $H(m+p)=\frac{\Gamma(m+p)}{\Gamma(m+p+\lambda+1)}$,(λ> 0, m $\geq$ 1),

then , for z= $re^{i\vartheta}$ and $0 < r < 1$

$$\int_0^{2\pi}\left|D_{\boldsymbol{z}}^{-\lambda}f(z)\right|^{\square}d\vartheta \leq \int_0^{2\pi}\left|D_{\boldsymbol{z}}^{-\lambda}f_n(z)\right|^{\square}d\vartheta\,.(\,\lambda > \,0, \square\, > \,0\,).$$

References

[1] W. G. Atshan and A. K. Wanas, On subclass of p-valent analytic functions with negative coefficients , Advances and Applications in Mathematical Sciences , 11 (5)(2012), 239-254.
[2] S. Bulut , On a class of analytic and multivalent functions with negative coefficients defined by Al-Oboudi differential operator, Studia Univ. "Babes-Bolyai", Mathematica,LV(4)(2010),115-130.
[3] S. P. Goyal,On aclass of multivalent functions defined by a generalized Ruscheweyh derivative operator, J. Indian Acad. Math. , 27(2)(2005). 439-456.
[4] Y. Komatu, On analytic prolongation of a family of operators, Mathematica (Cluj), 39(55)(1990) , 141-145.
[5] L. E. Littlewood , On inequalities in the theory of functions , Prec. London Math. Soc. , 23(1925),481-519.
[6] J. L. Liu , Certain convolution properties of multivalent analytic functions associated with a linear operator ,General Math. ,17(2)(2009) , 41-52.
[7] S. Najafzadeh , S. R. Kulkarni and G. Murugusundaramoorthy , Certain class of p-valent functions defined by Dziok-Srivastava linear operator , General Math. , 14(1)(2006), 65-76.
[8] H. M. Srivastava and S. Owa (Eds.),Current Topics in Analytic function Theory , World Scientific Publishing Company ,Singapore , 1992.