4. The relation between χ, γ and d_χ

In this section we derive the relation between the parameters χ, γ and d_χ of the Harary graph $H_{k,n}$.

Theorem 4.1
For any graph $H_{2r,n}$, we have $d_\chi(H_{2r,n}) = p$ such that $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

Proof:
Let V_i, $1 \leq i \leq k$ be the color class of $H_{2r,n}$. If $q = 0$, then $H_{2r,n}$ is uniquely p-colorable. By Theorem 2.2, $H_{2r,n}$ is uniquely p-colorable and therefore $d_\chi(H_{2r,n}) = \chi(H_{2r,n})) = p$. If $q \neq 0$, then $\chi(H_{2r,n}) = p + 1$. We claim that the color class V_{p+1} with q vertices is not a dominating set. Every v_i dominates v_j, with $i - r \leq j \leq i + r$ (where addition is taken to modulo n). There is no adjacent vertices v_j , with $n - r + 1 \leq j \leq n$ of v_1 has the color p+1 because $r(< p)$ is not a multiple of p. Since no vertex of V_{p+1} dominates v_1, dominating-χ-color number is $d_\chi(H_{2r,n}) = p$.

The following is then an immediate consequence of the Theorem 4.1. It gives the relation between the dominating -χ- color number and chromatic number of a Hararygraph $H_{2r,n}$.

Corollary 4.2
Let $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

$$\chi(H_{2r,n}) = \begin{cases} d_\chi(H_{2r,n}) \text{ if } q = 0 \\ d_\chi(H_{2r,n}) + 1 \text{ otherwise} \end{cases}$$

Theorem 4.3
For any Harary graph, $H_{k,n} = H_{2r,n}$ or $H_{2r+1,n}$, with $r \geq 1$, then $1 \leq \gamma(H_{k,n}) \leq s$ such that $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

Proof:
An obvious lower bound on the domination number is one. Since s is cardinality of maximal independent set, the domination number of $H_{k,n}$ is less than or equal to s.

The following result offers the relation between domination number, chromatic number and dominating -χ-color number of a Harary graph $H_{k,n}$ [7,8].

Theorem 4.4
For any Harary graph $H_{k,n}$ with $k \geq 2$, $d_\chi \leq \frac{n}{\gamma} \leq \chi + 1$

Proof:
Let c be the cardinality of the color class V_i with $1 \leq i \leq d_\chi$, which can dominate all other vertices of a graph. By definition of dominating χ color number, $c < \frac{n}{d_\chi}$

First, to show that $\gamma d_\chi < n$, assume to the contrary, that $\gamma d_\chi > n$. Then $c < \frac{n}{d_\chi} < \gamma$. Therefore the number of c with c <γ , vertices can dominate the graph. This is contradiction.

To show that $n \leq \gamma(\chi + 1)$, assume to the contrary, that $n > \gamma(\chi + 1)$. And let s as defined in the Theorem 4.3. Then n /(χ+1) >γ . Since $\leq r + 1, s > \gamma$. By Theorem 4.3, which contradicts the fact that $< \gamma$.

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