The Exp$_{(-\varphi(\xi))}$-Expansion Method and its Application for Solving Nonlinear Evolution Equations

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Abstract: ***In this paper, we employ the exp*** $(-\varphi(\xi))$ ***method to find the exact traveling wave solutions involving parameters of a nonlinear the system of shallow water wave equations. Comparison between this two method and another method which have been solved it. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. It is shown that the proposed methods provides a more powerful mathematical tool for constructing exact traveling wave solutions for many other nonlinear evolution equations.***

Keywords: The exp $(-\varphi(\xi))$ -expansion method; The system of shallow water wave equations; Traveling wave solutions; Solitary wave solutions; Kink-anti-kink shaped.

AMS subject classifications: 35A05, 35A20, 65K99, 65Z05, 76R50, 70K70

1. Introduction

The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Exact solutions for these equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, Optics, Plasma physics and so on. Recently many new approaches for finding these solutions have been proposed, for example, tanh - sech method [2]-[4], extended tanh - method [5]-[7], sine - cosine method [8]-[10], homogeneous balance method [11, 12], the modified simple equation [13] ,F-expansion method [14]-[15], exp-function method [16, 17], trigonometric function series method [18], $(\frac{G'}{G})-$ expansion method [19]-[22], Jacobi elliptic function method [23]-[26], The exp $(-\varphi(\xi))$ -expansion method[27]-[29] and so on.

The objective of this article is to apply the exp $(-\varphi(\xi))$ -expansion method for finding the exact traveling wave solution of the system of shallow water wave equations which play an important role in biology and mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of The exp $(-\varphi(\xi))$ -expansion method. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. Description of Method

Consider the following nonlinear evolution equation

$$F(u,u_t,u_x,u_{tt},u_{xx},....)=0, \qquad (1)$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method

Step 1. We use the wave transformation

$$u(x,t)=u(\xi), \qquad \xi=x-ct, \qquad (2)$$

where c is a positive constant, to reduce Eq.(1) to the following ODE:

$$P(u,u',u'',u''',.....)=0, \qquad (3)$$

where P is a polynomial in $u(\xi)$ and its total derivatives,while $'=\frac{d}{d\xi}'$.

Step 2. Suppose that the solution of ODE(3) can be expressed by a polynomial in $exp(-\varphi(\xi))$ as follows

$$u(\xi)=a_m\big(exp(-\varphi(\xi))\big)^m+....., \quad a_m\neq 0, \qquad (4)$$

where $\varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi)=exp(-\varphi(\xi))+\mu exp(\varphi(\xi))+\lambda, \qquad (5)$$

the solutions of ODE (5) are

when $\lambda^2-4\mu>0, \mu\neq 0,$

$$\varphi(\xi)=ln\left(\frac{-\sqrt{\lambda^2-4\mu}\,tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+C_1)\right)-\lambda}{2\mu}\right), \qquad (6)$$

when $\lambda^2-4\mu>0, \mu=0,$

$$\varphi(\xi)=-ln\left(\frac{\lambda}{exp(\lambda(\xi+C_1))-1}\right), \qquad (7)$$

when $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0,$

$$\varphi(\xi)=ln\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right), \qquad (8)$$

when $\lambda^2-4\mu=0, \mu=0, \lambda=0,$

$$\varphi(\xi)=ln(\xi+C_1), \qquad (9)$$

when $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = ln\left(\frac{\sqrt{4\mu - \lambda^2}\, tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C_1)\right) - \lambda}{2\mu}\right), \tag{10}$$

where $a_m, \ldots, \lambda, \mu$ are constants to be determined later,

Step 3. Substitute Eq.(4) along Eq.(5) into Eq.(3) and collecting all the terms of the same power $exp(-m\varphi(\xi))$, $(m = 0,1,2,3,\ldots)$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i.

Step 4. substituting these values and the solutions of Eq.(5) into Eq.(4) we obtain the exact solutions of Eq.(3).

3. The System of Shallow Water Wave Equations

Here, we will apply the exp (-()) -expansion method described in sec.2 to find the exact traveling wave solutions and then the solitary wave solutions for the system of shallow water wave equations.

We first consider the system of the shallow water wave equation[30] in order to demonstrate the $exp(-\varphi(\xi))$ -expansion method

$$\{u_t + (uv)_x + v_{xxx} = 0, v_t + u_x + vv_x = 0. \tag{11}$$

We use the wave transformation $u(x,t) = u(\xi), \xi = x - ct$ to reduce Eqs.(11) to the following nonlinear system of ordinary differential equations:

$$\{-cu' + vu' + uv' + v''' = 0, u' - cv' + vv' = 0, \tag{12}$$

where by integrating once the second equation with zero constant of integration, we find

$$u = cv - \frac{v^2}{2}. \tag{13}$$

substituting Eq.(13) into the first equation of Eq.(12) we obtain

$$v''' + (3cv - \frac{3v^2}{2} - c^2)v' = 0. \tag{14}$$

Integrating Eq.(14) with zero constant of integration, we find

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2 v = 0. \tag{15}$$

Balancing v'' and v^3 in Eq.(15) yields, $N + 2 = 3N \Rightarrow N = 1$. This suggests the choice of $v(\xi)$ in Eq.(15) as

$$v(\xi) = a_0 + a_1 exp(-\varphi(\xi)), \tag{16}$$

where $\varphi(\xi)$ satisfies the second order linear ODE (5).

Substituting Eq.(16) and its derivative by using (5) in Eq.(15) and rationalization of the resultant expression with respect to the powers of $exp(-\varphi(\xi))$ yield the following set of algebraic equations

$$exp(-3\varphi(\xi)) : 2a_1 - \frac{1}{2}a_1^{\,3} = 0, \tag{17}$$

$$exp(-2\varphi(\xi)) : 3\lambda a_1 + \frac{3}{2}ca_1^{\,2} - \frac{3}{2}a_0 a_1^{\,2} = 0, \tag{18}$$

$$exp(-1\varphi(\xi)) : a_1(\lambda^2 + 2\mu) + 3ca_0 a_1 - \frac{3}{2}a_0^{\,2} a_1 - c^2 a_1 = 0, \tag{19}$$

$$exp(-0\varphi(\xi)) : \lambda\mu a_1 + \frac{3}{2}ca_0^{\,2} - \frac{1}{2}a_0^{\,3} - c^2 a_0 = 0. \tag{20}$$

On solving the above set of algebraic equations, we have

Case 1.

$$c = c, \lambda = \lambda, \mu = \pm\frac{1}{2}\lambda a_0 - \frac{1}{4}a_0^2, a_0 = c \pm \lambda, a_1 = \pm 2,$$

Now, with the help of (6)-(10) and (16) , we obtain the following types of traveling wave solutions of Eq.(15).

$$v(\xi) = c \pm \lambda \pm 2(exp(-\varphi(\xi))),$$

$$u(\xi) = c(c \pm \lambda \pm 2(exp(-\varphi(\xi)))) - \frac{1}{2}(c \pm \lambda \pm 2(exp(-\varphi(\xi))))^2.$$

So that by using Eq.(6)-(10) the exact solution of Eq.(15) can be written in the form :

When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$v(\xi) = c \pm \lambda \pm \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu}\, tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C_1)\right) - \lambda}, \tag{21}$$

$$u(\xi) = c\left(c \pm \lambda \pm \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu}\, tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C_1)\right) - \lambda}\right)$$

$$-\frac{1}{2}\left(c\pm\lambda\pm\frac{4\mu}{-\sqrt{\lambda^2-4\mu}\,tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\xi+C_1\right)\right)-\lambda}\right)^2. \quad (22)$$

When $\lambda^2-4\mu>0,\mu=0,$

$$v(\xi)=c\pm\lambda\pm\frac{2\lambda}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}, \quad (23)$$

$$u(\xi)=c\left(c\pm\lambda\pm\frac{2\lambda}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}\right)$$
$$-\frac{1}{2}\left(c\pm\lambda\pm\frac{2\lambda}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}\right)^2. \quad (24)$$

When $\lambda^2-4\mu=0,\mu\neq0,\lambda\neq0,$

$$v(\xi)=c\pm\lambda\mp\frac{\lambda^2\left(\xi+C_1\right)}{\left(\lambda\left(\xi+C_1\right)+2\right)}, \quad (25)$$

$$u(\xi)=c\left(c\pm\lambda\mp\frac{\lambda^2\left(\xi+C_1\right)}{\left(\lambda\left(\xi+C_1\right)+2\right)}\right)$$
$$-\frac{1}{2}\left(c\pm\lambda\mp\frac{\lambda^2\left(\xi+C_1\right)}{\left(\lambda\left(\xi+C_1\right)+2\right)}\right)^2. \quad (26)$$

When $\lambda^2-4\mu=0,\mu=0,\lambda=0,$

$$v(\xi)=c\pm\lambda\pm\frac{2}{\left(\xi+C_1\right)}, \quad (27)$$

$$u(\xi)=c\left(c\pm\lambda\pm\frac{2}{\left(\xi+C_1\right)}\right)-\frac{1}{2}\left(c\pm\lambda\pm\frac{2}{\left(\xi+C_1\right)}\right)^2. \quad (28)$$

When $\lambda^2-4\mu<0,$

$$v(\xi)=c\pm\lambda\pm\frac{4\mu}{\sqrt{4\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\xi+C_1\right)\right)-\lambda}, \quad (29)$$

$$u(\xi)=c\left(c\pm\lambda\pm\frac{4\mu}{\sqrt{4\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\xi+C_1\right)\right)-\lambda}\right) \quad (30)$$

$$-\frac{1}{2}\left(c\pm\lambda\pm\frac{4\mu}{\sqrt{4\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\xi+C_1\right)\right)-\lambda}\right)^2. \quad (31)$$

[Eq.(21)] [Eq.(23)] [Eq.(25)]

Figure 1: Solitary traveling wave solution of Eqs.(21), (23)and (25)

[Eq.(27)] [Eq.(29)]

Figure 2: Solitary traveling wave solution of Eqs. (27) and (29)

4. Conclusion

The exp(-()) -expansion method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for the system of shallow water wave equations which have been constructed using the exp(-()) -expansion method. Fig.[1 and 2] show the solitary wave solution of the system of shallow water wave equations. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the system of shallow water wave equations are new and different from those obtained in [30]. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations.

References

[1] M. J. Ablowitz, H. Segur, Solitions and Inverse Scattering Transform, SIAM, Philadelphia 1981.

[2] W. Malfliet, Solitary wave solutions of nonlinear wave equation, Am. J. Phys., 60 (1992) 650-654.

[3] W. Malfliet, W. Hereman, The tanh method: Exact solutions of nonlinear evolution and wave equations, Phys.Scr., 54 (1996) 563-568.

[4] A. M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, Appl. Math. Comput., 154 (2004) 714-723.

[5] S. A. EL-Wakil, M.A.Abdou, New exact travelling wave solutions using modified extented tanh-function method, Chaos Solitons Fractals, 31 (2007) 840-852.

[6] E. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

[7] Mahmoud A.E. Abdelrahman, Emad H. M. Zahran Mostafa M.A. Khater, Exact Traveling Wave Solutions for Modified Liouville Equation Arising in Mathematical Physics and Biology, International Journal of Computer Applications (0975 8887) Volume 112 - No. 12, February (2015).

[8] A. M. Wazwaz, Exact solutions to the double sinh-Gordon equation by the tanh method and a variable separated ODE. method, Comput. Math. Appl., 50 (2005) 1685-1696.

[9] A. M. Wazwaz, A sine-cosine method for handling nonlinear wave equations, Math. Comput. Modelling, 40 (2004) 499-508.
[10] C. Yan, A simple transformation for nonlinear waves, Phys. Lett. A 224 (1996) 77-84.
[11] E. Fan, H.Zhang, A note on the homogeneous balance method, Phys. Lett. A 246 (1998) 403-406.
[12] M. L. Wang, Exct solutions for a compound KdV-Burgers equation, Phys. Lett. A 213 (1996) 279-287.
[13] Emad H. M. Zahran and Mostafa M.A. Khater, The modified simple equation method and its applications for solving some nonlinear evolutions equations in mathematical physics, (Jokull journal- Vol. 64. Issue 5 - May 2014).
[14] Y. J. Ren, H. Q. Zhang, A generalized F−expansion method to find abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equation, Chaos Solitons Fractals, 27 (2006) 959-979.
[15] J. L. Zhang, M. L. Wang, Y. M. Wang, Z. D. Fang, The improved F-expansion method and its applications, Phys.Lett.A 350 (2006) 103-109.
[16] J. H. He, X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons Fractals 30 (2006) 700-708.
[17] H. Aminikhad, H. Moosaei, M. Hajipour, Exact solutions for nonlinear partial differential equations via Exp-function method, Numer. Methods Partial Differ. Equations, 26 (2009) 1427-1433.
[18] Z. Y. Zhang, New exact traveling wave solutions for the nonlinear Klein-Gordon equation, Turk. J. Phys., 32 (2008) 235-240.
[19] M. L. Wang, J. L. Zhang, X. Z. Li, The $(\frac{G'}{G})$ -expansion method and travelling wave solutions of nonlinear evolutions equations in mathematical physics, Phys. Lett. A 372 (2008) 417-423.
[20] S. Zhang, J. L. Tong, W.Wang, A generalized $(\frac{G'}{G})$ -expansion method for the mKdv equation with variable coefficients, Phys. Lett. A 372 (2008) 2254-2257.
[21] E. M. E. Zayed and K. A. Gepreel, The $(\frac{G'}{G})$ -expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics, J. Math. Phys., 50 (2009) 013502-013513.
[22] E.H.M.Zahran and mostafa M.A. khater, Exact solutions to some nonlinear evolution equations by the $(\frac{G'}{G})$ -expansion method equations in mathematical physics, J $\ddot{o}$ kull Journal, Vol. 64, No. 5; May 2014.
[23] C. Q. Dai , J. F. Zhang, Jacobian elliptic function method for nonlinear differential difference equations, Chaos Solutions Fractals, 27 (2006) 1042-1049.
[24] E. Fan , J .Zhang, Applications of the Jacobi elliptic function method to special-type nonlinear equations, Phys. Lett. A 305 (2002) 383-392.
[25] S. Liu, Z. Fu, S. Liu, Q.Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A 289 (2001) 69-74.
[26] Emad H. M. Zahran and Mostafa M.A. Khater, Exact Traveling Wave Solutions for the System of Shallow Water Wave Equations and Modified Liouville Equation Using Extended Jacobian Elliptic Function Expansion Method, American Journal of Computational Mathematics (AJCM) Vol.4 No.5 (2014).
[27] Nizhum Rahman, Md. Nur Alam, Harun-Or-Roshid, Selina Akter and M. Ali Akbar, Application of $exp(-\varphi(\xi))$ expansion method to find the exact solutions of Shorma-Tasso-Olver Equation, African Journal of Mathematics and Computer Science Research Vol. 7(1), pp. 1-6, February, 2014.
[28] Rafiqul Islam, Md. Nur Alam, A.K.M. Kazi Sazzad Hossain, Harun-Or-Roshid and M. Ali Akbar, Traveling Wave Solutions of Nonlinear Evolution Equations via Exp $(-\phi(\xi))$ -Expansion Method, Global Journal of Science Frontier Research Mathematics and Decision Sciences. Volume 13 Issue 11 Version 1.0 Year 2013.
[29] Mahmoud A.E. Abdelrahman, Emad H. M. Zahran Mostafa M.A. Khater, Exact traveling wave solutions for power law and Kerr law non linearity using the exp $(-\phi(\xi))$ -expansion method . (GJSFR Volume 14-F Issue 4 Version 1.0).
[30] $\dot{I}$ hsan Timucin Dolapci, Ahmet Yildirim, Some exact solutions to the generalized Korteweg-de Vries equation and the system of shallow water wave equation, Nonlinear analysis, Modeling and control,(2013), Vol. 18, No. 1,27-36.