

The $\exp(-\varphi(\xi))$ -Expansion Method and its Application for Solving Nonlinear Evolution Equations

Mahmoud A. E. Abdelrahman¹, Mostafa M. A. Khater²

^{1, 2}Mansoura University, Department of Mathematics, Faculty of Science, 35516 Mansoura, Egypt

Abstract: In this paper, we employ the $\exp(-\varphi(\xi))$ method to find the exact traveling wave solutions involving parameters of a nonlinear the system of shallow water wave equations. Comparison between this two method and another method which have been solved it. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. It is shown that the proposed methods provides a more powerful mathematical tool for constructing exact traveling wave solutions for many other nonlinear evolution equations.

Keywords: The $\exp(-\varphi(\xi))$ -expansion method; The system of shallow water wave equations; Traveling wave solutions; Solitary wave solutions; Kink-anti-kink shaped.

AMS subject classifications: 35A05, 35A20, 65K99, 65Z05, 76R50, 70K70

1. Introduction

The nonlinear partial differential equations of mathematical physics are major subjects in physical science [1]. Exact solutions for these equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, Optics, Plasma physics and so on. Recently many new approaches for finding these solutions have been proposed, for example, tanh - sech method [2]-[4], extended tanh - method [5]-[7], sine - cosine method [8]-[10], homogeneous balance method [11, 12], the modified simple equation [13], F-expansion method [14]-[15], exp-function method [16, 17], trigonometric function series method [18],

$\left(\frac{G'}{G}\right)$ - expansion method [19]-[22], Jacobi elliptic function method [23]-[26], The $\exp(-\varphi(\xi))$ -expansion method [27]-[29] and so on.

The objective of this article is to apply the $\exp(-\varphi(\xi))$ -expansion method for finding the exact traveling wave solution of the system of shallow water wave equations which play an important role in biology and mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of The $\exp(-\varphi(\xi))$ -expansion method. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. Description of Method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0, \quad (1)$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method

Step 1. We use the wave transformation

$$u(x, t) = u(\xi), \quad \xi = x - ct, \quad (2)$$

where c is a positive constant, to reduce Eq.(1) to the following ODE:

$$P(u, u', u'', u''', \dots) = 0, \quad (3)$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.

Step 2. Suppose that the solution of ODE(3) can be expressed by a polynomial in $\exp(-\varphi(\xi))$ as follows

$$u(\xi) = a_m (\exp(-\varphi(\xi)))^m + \dots, \quad a_m \neq 0, \quad (4)$$

where $\varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \quad (5)$$

the solutions of ODE (5) are

when $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$\varphi(\xi) = \ln \left[\frac{-\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right], \quad (6)$$

when $\lambda^2 - 4\mu > 0, \mu = 0$,

$$\varphi(\xi) = -\ln \left(\frac{\lambda}{\exp(\lambda(\xi + C_1)) - 1} \right), \quad (7)$$

when $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$\varphi(\xi) = \ln \left(-\frac{2(\lambda(\xi + C_1) + 2)}{\lambda^2(\xi + C_1)} \right), \quad (8)$$

when $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$\varphi(\xi) = \ln(\xi + C_1), \quad (9)$$

when $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = \ln \left[\frac{\sqrt{4\mu - \lambda^2} \tan \left(\frac{\sqrt{4\mu - \lambda^2}}{2} (\xi + C_1) \right) - \lambda}{2\mu} \right], \quad (10)$$

where a_m, \dots, λ, μ are constants to be determined later,

Step 3. Substitute Eq.(4) along Eq.(5) into Eq.(3) and collecting all the terms of the same power $\exp(-m\varphi(\xi))$, ($m = 0, 1, 2, 3, \dots$) and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i .

Step 4. substituting these values and the solutions of Eq.(5) into Eq.(4) we obtain the exact solutions of Eq.(3).

3. The System of Shallow Water Wave Equations

Here, we will apply the $\exp(-\varphi)$ -expansion method described in sec.2 to find the exact traveling wave solutions and then the solitary wave solutions for the system of shallow water wave equations.

We first consider the system of the shallow water wave equation[30] in order to demonstrate the $\exp(-\varphi(\xi))$ -expansion method

$$\{u_t + (uv)_x + v_{xxx} = 0, v_t + u_x + vv_x = 0. \quad (11)$$

We use the wave transformation $u(x, t) = u(\xi), \xi = x - ct$ to reduce Eqs.(11) to the following nonlinear system of ordinary differential equations:

$$\{-cu' + vu' + uv' + v''' = 0, u' - cv' + vv' = 0, \quad (12)$$

where by integrating once the second equation with zero constant of integration, we find

$$u(\xi) = c(c \pm \lambda \pm 2(\exp(-\varphi(\xi)))) - \frac{1}{2}(c \pm \lambda \pm 2(\exp(-\varphi(\xi))))^2.$$

So that by using Eq.(6)-(10) the exact solution of Eq.(15) can be written in the form :

When $\lambda^2 - 4\mu > 0, \mu \neq 0$,

$$v(\xi) = c \pm \lambda \pm \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda}, \quad (21)$$

$$u(\xi) = c \left(c \pm \lambda \pm \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} (\xi + C_1) \right) - \lambda} \right)$$

$$u = cv - \frac{v^2}{2}. \quad (13)$$

substituting Eq.(13) into the first equation of Eq.(12) we obtain

$$v''' + (3cv - \frac{3v^2}{2} - c^2)v' = 0. \quad (14)$$

Integrating Eq.(14) with zero constant of integration, we find

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2v = 0. \quad (15)$$

Balancing v'' and v^3 in Eq.(15) yields, $N + 2 = 3N \Rightarrow N = 1$. This suggests the choice of $v(\xi)$ in Eq.(15) as

$$v(\xi) = a_0 + a_1 \exp(-\varphi(\xi)), \quad (16)$$

where $\varphi(\xi)$ satisfies the second order linear ODE (5).

Substituting Eq.(16) and its derivative by using (5) in Eq.(15) and rationalization of the resultant expression with respect to the powers of $\exp(-\varphi(\xi))$ yield the following set of algebraic equations

$$\exp(-3\varphi(\xi)): 2a_1 - \frac{1}{2}a_1^3 = 0, \quad (17)$$

$$\exp(-2\varphi(\xi)): 3\lambda a_1 + \frac{3}{2}ca_1^2 - \frac{3}{2}a_0a_1^2 = 0, \quad (18)$$

$$\exp(-1\varphi(\xi)): a_1(\lambda^2 + 2\mu) + 3ca_0a_1 - \frac{3}{2}a_0^2a_1 - c^2a_1 = 0, \quad (19)$$

$$\exp(-0\varphi(\xi)): \lambda\mu a_1 + \frac{3}{2}ca_0^2 - \frac{1}{2}a_0^3 - c^2a_0 = 0. \quad (20)$$

On solving the above set of algebraic equations, we have

Case 1.

$$c = c, \lambda = \lambda, \mu = \pm \frac{1}{2}\lambda a_0 - \frac{1}{4}a_0^2, a_0 = c \pm \lambda, a_1 = \pm 2,$$

Now, with the help of (6)-(10) and (16), we obtain the following types of traveling wave solutions of Eq.(15).

$$v(\xi) = c \pm \lambda \pm 2(\exp(-\varphi(\xi))),$$

$$-\frac{1}{2} \left(c \pm \lambda \pm \frac{4\mu}{-\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C_1)\right) - \lambda} \right)^2 \quad (22)$$

When $\lambda^2 - 4\mu > 0, \mu = 0$,

$$v(\xi) = c \pm \lambda \pm \frac{2\lambda}{\exp(\lambda(\xi + C_1)) - 1}, \quad (23)$$

$$u(\xi) = c \left(c \pm \lambda \pm \frac{2\lambda}{\exp(\lambda(\xi + C_1)) - 1} \right) - \frac{1}{2} \left(c \pm \lambda \pm \frac{2\lambda}{\exp(\lambda(\xi + C_1)) - 1} \right)^2 \quad (24)$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$v(\xi) = c \pm \lambda \mp \frac{\lambda^2(\xi + C_1)}{(\lambda(\xi + C_1) + 2)}, \quad (25)$$

$$u(\xi) = c \left(c \pm \lambda \mp \frac{\lambda^2(\xi + C_1)}{(\lambda(\xi + C_1) + 2)} \right) - \frac{1}{2} \left(c \pm \lambda \mp \frac{\lambda^2(\xi + C_1)}{(\lambda(\xi + C_1) + 2)} \right)^2 \quad (26)$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$v(\xi) = c \pm \lambda \pm \frac{2}{(\xi + C_1)}, \quad (27)$$

$$u(\xi) = c \left(c \pm \lambda \pm \frac{2}{(\xi + C_1)} \right) - \frac{1}{2} \left(c \pm \lambda \pm \frac{2}{(\xi + C_1)} \right)^2 \quad (28)$$

When $\lambda^2 - 4\mu < 0$,

$$v(\xi) = c \pm \lambda \pm \frac{4\mu}{\sqrt{4\mu - \lambda^2} \tanh\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C_1)\right) - \lambda}, \quad (29)$$

$$u(\xi) = c \left(c \pm \lambda \pm \frac{4\mu}{\sqrt{4\mu - \lambda^2} \tanh\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C_1)\right) - \lambda} \right) \quad (30)$$

$$-\frac{1}{2} \left(c \pm \lambda \pm \frac{4\mu}{\sqrt{4\mu - \lambda^2} \tanh\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C_1)\right) - \lambda} \right)^2 \quad (31)$$

[Eq.(21)] [Eq.(23)] [Eq.(25)]

Figure 1: Solitary traveling wave solution of Eqs.(21), (23) and (25)

[Eq.(27)] [Eq.(29)]

Figure 2: Solitary traveling wave solution of Eqs. (27) and (29)

4. Conclusion

The $\exp(-\cdot)$ -expansion method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for the system of shallow water wave equations which have been constructed using the $\exp(-\cdot)$ -expansion method. Fig.[1 and 2] show the solitary wave solution of the system of shallow water wave equations. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the system of shallow water wave equations are new and different from those obtained in [30]. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations.

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