\( \pi \beta \)-Normal Spaces in Topological Spaces

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Abstract: The aim of this paper is to introduce a new class of normal spaces called \( \pi \beta \)-normal spaces, by using \( \beta \)-open sets. We prove that \( \pi \beta \)-normality is a topological property and it is a hereditary property with respect to \( \pi \)-open, \( \beta \)-closed subspaces. Further we obtain a characterization and preservation theorems for \( \pi \beta \)-normal spaces.

Keywords: regular closed, \( \pi \)-closed, \( \pi \beta \)-closed, and \( \beta \)-open sets; pre \( \beta \)-closed, \( \pi \)-continuous, \( \pi \beta \)-continuous, \( \pi \)-irresolute, \( \pi \beta \)-irresolute and almost \( \beta \)-irresolute functions; \( \pi \beta \)-normal spaces

1. Introduction

In 1970, Levine [7] defined generalized closed sets in topological spaces. In 1989, Nour [9] introduced the notion of \( p \)-normal spaces and obtained characterization and preservation theorems for \( p \)-normal spaces. In 1990, Mahmoud and Monsef [8] introduced the concept of \( \pi \)-normal spaces. In 1995, Dontchev [5] introduced a new class of sets called \( g \beta \)-closed sets. In 2010, Tahiliani [10] introduced the notion of \( \pi \beta \)-closed sets and its properties are studied. Recently, Thanh and Thinh [12] introduced the notion of \( \pi \beta \)-normal spaces and prove that \( \pi \beta \)-normality is a topological property and it is a hereditary property with respect to \( \pi \)-open, \( \pi \beta \)-closed subspaces.

In this paper, we introduce and study a new class of normal spaces called \( \pi \beta \)-normal spaces by using \( \beta \)-open sets. We prove that \( \pi \beta \)-normality is a topological property and it is a hereditary property with respect to \( \pi \)-open, \( \pi \beta \)-closed subspaces. Further we obtain a characterization and preservation theorems for \( \pi \beta \)-normal spaces.

2. Preliminaries

Throughout this paper, spaces \((X, \tau)\), \((Y, \sigma)\), and \((Z, \gamma)\) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let \( A \) be a subset of a space \( X \). The closure of \( A \) and interior of \( A \) are denoted by \( \text{cl}(A) \) and \( \text{int}(A) \) respectively. A subset \( A \) is said to be \( \text{regular open} \) \([6]\) (resp. \( \text{regular closed} \) \([6]\)) if \( A = \text{cl}(\text{int}(A)) \) (resp. \( A = \text{cl}(A) \)). The finite union of regular open sets is said to be \( \pi \)-open \([15]\). The complement of a \( \pi \)-open set is said to be \( \pi \)-closed \([15]\). A is said to be \( \beta \)-open \([1]\) if \( A \subset \text{cl}(\text{int}(A)) \). The family of all \( \beta \)-open sets of \( X \) is denoted by \( \beta O(X) \). The \( \beta \)-closure of \( A \), denoted by \( \beta \text{cl}(A) \), is defined as union of all \( \beta \)-open sets contained in \( A \). The family of all \( \beta \)-open (resp. \( \beta \)-closed) sets of \( X \) is denoted by \( \beta O(X) \) (resp. \( \beta C(X) \)).

2.1 Definition

A subset \( A \) of a space \( X \) is said to be
1. \( \text{generalized closed} \) (briefly \( g \)-closed) \([7]\) if \( \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \in \tau \).
2. \( \text{generalized } \beta \text{-closed} \) (briefly \( g \beta \)-closed) \([5]\) if \( \beta \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \in \tau \).
3. \( \pi \beta \)-closed \([10]\) if \( \beta \text{cl}(A) \subset U \) whenever \( A \subset U \) and \( U \in \pi \text{-open in } X \).
4. \( g \)-open (resp. \( g \beta \)-open, \( \pi \beta \)-open) if the complement of \( A \) is \( g \)-closed (resp. \( g \beta \)-closed, \( \pi \beta \)-closed).

The intersection of all \( \pi \beta \)-closed sets containing \( A \) is called \( \pi \beta \)-closure of \( A \), and is denoted by \( \pi \beta \text{cl}(A) \). The \( \pi \beta \)-interior of \( A \), denoted by \( \pi \beta \text{int}(A) \), is defined as union of all \( \pi \beta \)-open sets contained in \( A \). The family of all \( \pi \beta \)-open (resp. \( \pi \beta \)-closed) sets of \( X \) is denoted by \( \pi \beta O(X) \) (resp. \( \pi \beta C(X) \)).

2.2 Definition

A space \( X \) is said to be \( \beta \)-normal \([8]\) (resp. \( p \)-normal \([9]\)) if for every pair of disjoint closed subsets \( A, B \) of \( X \), there exist disjoint \( \beta \)-open (resp. \( p \)-open) sets \( U, V \) of \( X \) such that \( A \subset U \) and \( B \subset V \).

2.3 Definition

A space \( X \) is said to be \( \pi \beta \)-normal \([14]\) (resp. \( \pi p \)-normal \([11]\)) if for every pair of disjoint closed subsets \( A, B \) of \( X \), one of which is \( \pi \)-closed, there exist disjoint \( \beta \)-open (resp. \( p \)-open) sets \( U, V \) of \( X \) such that \( A \subset U \) and \( B \subset V \).

2.4 Definition

A subset \( A \) of a space \( X \) is said to be a \( \beta \)-neighborhood \([8]\) of \( x \) if there exists a \( \beta \)-open set \( U \) such that \( x \in U \subset A \).

2.5 Definition

A function \( f : X \to Y \) is said to be
(a) \( \text{regular open} \) \([13]\) if \( f(U) \) is regular open in \( Y \) for every open set \( U \) in \( X \).
(b) **π-continuous** [4] if $f^{-1}(F)$ is π-closed in $X$ for each closed set $F$ in $Y$.

(c) **pre-β-closed** [8] (resp. **pre β–open** [8]) $f(F)$ is β-closed (resp. β-open) set in for every β–closed (resp. β–open) set $F$ in $X$.

(d) **πβ-continuous** [10] if $f^{-1}(F)$ is πβ-closed in $X$ for every closed set $F$ in $Y$.

(e) **πβ-irresolute** [10] if $f^{-1}(F)$ is πβ-closed in $X$ for every πβ-closed set $F$ in $Y$.

(f) **almost β-irresolute** [8] if for each $x \in X$ and β–neighborhood $V$ of $f(x)$ in $Y$, $\beta cl(f^{-1}(V))$ is neighborhood of $x$ in $X$.

3. **πβ-Normal Spaces**

In this section, we introduce the notion of πβ–normal space and study some property of it. First, we begin with the following definitions and examples.

3.1 Definition

A space $X$ is said to be **πβ–normal** (resp. **πgp–normal** [12]) if for every pair of disjoint πβ-closed (resp. πgp-closed) subsets $H$ and $K$ of $X$, there exist disjoint β–open (resp. p-open) sets $U$, $V$ of $X$ such that $H \subseteq U$ and $K \subseteq V$.

Clearly, from above definitions, we have the following diagram:

$$
\begin{array}{c}
\pi gp - normality \Rightarrow p - normality \Rightarrow \pi p - normality \\
\downarrow \quad \downarrow \\
\pi β - normality \Rightarrow β - normality \Rightarrow πβ - normality
\end{array}
$$

Where none of the above implications is reversible as can be seen from the following examples:

3.2 Example

We consider the topology $\tau = \{\emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$ on the set $X = \{a, b, c, d\}$. Then, the space $X$ is p-normal as well as β–normal. But it is neither πgp–normal nor πβ–normal.

3.3 Example

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then the space $X$ is β–normal as well as πβ–normal but it is not p-normal.

3.4 Example

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. The pair of disjoint closed subsets of $X$ are $A = \{a\}$ and $B = \{c\}$. Also $U = \{a, b\}$ and $V = \{c, d\}$ are β–open sets such that $A \subseteq U$ and $B \subseteq V$. Hence $X$ is β–normal as well as πβ–normal.

3.5 Example

Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then, the space $X$ is β–normal.

3.6 Theorem

For a topological space $X$, the following are equivalent:

(a) $X$ is πβ–normal.

(b) For every pair of disjoint πβ-closed subsets $U$ and $V$ of $X$ whose union is $X$, there exist β–closed subsets $G$ and $H$ of $X$ such that $G \subseteq U$, $H \subseteq V$ and $G \cap H = X$.

(c) For every πβ-closed set $A$ and every πβ-open set $B$ in $X$ such that $A \subseteq B$, there exists a β–open subset $V$ of $X$ such that $A \subseteq V \subseteq \beta cl(V) \subseteq B$.

(d) For every pair of disjoint πβ-closed subsets $A$ and $B$ of $X$, there exists a β–open subset $V$ of $X$ such that $A \subseteq V \subseteq \beta cl(V) \subseteq B$.

(e) For every pair of disjoint πβ-closed subsets $A$ and $B$ of $X$, there exist disjoint β–open subsets $U$ and $V$ of $X$ such that $A \subseteq U$, $B \subseteq V$ and $\beta cl(U) \cap \beta cl(V) = \emptyset$.

Proof

(a) $\Rightarrow$ (b) Let $U$ and $V$ be any disjoint πβ-closed subsets of a πβ–normal space $X$ such that $U \cup V = X$. Then, $X \setminus U$ and $X \setminus V$ are disjoint πβ-closed subsets of $X$. By πβ-normality of $X$, there exist disjoint β–open subsets $U_1$ and $V_1$ of $X$ such that $X \setminus U \subseteq U_1$ and $X \setminus V \subseteq V_1$. Let $G = X \setminus U_1$ and $H = X \setminus V_1$. Then, $G$ and $H$ are β–closed subsets in $X$ such that $G \cap H = X$.

(b) $\Rightarrow$ (c) Let $A$ be a πβ–closed and $B$ be πβ–open subsets of $X$ such that $A \subseteq B$. Then, $A \cap (X \setminus B) = \emptyset$. Thus, $X \setminus A$ and $B$ are πβ–open subsets of $X$ such that $(X \setminus A) \cup B = X$. By the Part (b), there exist β–closed subsets $G$ and $H$ of $X$ such that $G \subseteq (X \setminus A)$, $H \subseteq B$ and $G \cup H = X$. Thus, we obtain that $A \subseteq (X \setminus G) \subseteq H \subseteq B$. Let $V = X \setminus G$. Then $V$ is β–open subset of $X$ and $\beta cl(V) \subseteq H$ as $H$ is β–closed set in $X$. Therefore, $A \subseteq V \subseteq \beta cl(V) \subseteq B$.

(c) $\Rightarrow$ (d) Let $A$ and $B$ be disjoint πβ-closed subset of $X$. Then $A \subseteq X \setminus B$, where $X \setminus B$ is β–open. By the Part (c), there exists a β–closed subset $U$ of $X$ such that $A \subseteq U \subseteq \beta cl(U) \subseteq X \setminus B$. Thus, $\beta cl(U) \cap B = \emptyset$.

(d) $\Rightarrow$ (e) Let $A$ and $B$ be any disjoint πβ-closed subset of $X$. Then by the part (d), there exist a β–closed set $U$ containing $A$ such that $\beta cl(U) \cap B = \emptyset$. Since $\beta cl(U)$ is β–closed, it is πβ-closed. Thus $\beta cl(U)$ and $B$ are disjoint πβ-closed subsets of $X$. Again by the part (d), there exists a β–open set $V$ in $X$ such that $B \subseteq V$ and $\beta cl(U) \cap \beta cl(V) = \emptyset$.

(e) $\Rightarrow$ (a) Let $A$ and $B$ be any disjoint πβ-closed subsets of $X$. Then by the part (e), there exist β–closed sets $U$ and $V$ such that $A \subseteq U$, $B \subseteq V$ and $\beta cl(U) \cap \beta cl(V) = \emptyset$. Therefore, we obtain that $U \cap V = \emptyset$ and hence $X$ is πβ–normal.
3.7 Lemma

(a) The image of \( \beta \)-open subset under an open continuous function is \( \beta \)-open.
(b) The inverse image of \( \beta \)-open (resp. \( \beta \)-closed) subset under an open continuous function is \( \beta \)-open (resp. \( \beta \)-closed) subset.

3.8 Lemma [12]

The image of a regular open subset under open and closed continuous function is regular open subset.

3.9 Proposition [12]

The image of a \( \pi \)-open subset under open and closed continuous function is \( \pi \)-open set.

3.10 Proposition

If \( f : X \to Y \) be an open and closed continuous bijection function and \( A \) be a \( \pi \beta \)-closed set in \( Y \), then \( f^{-1}(A) \) is \( \pi \beta \)-closed set in \( X \).

Proof. Let \( A \) be a \( \pi \beta \)-closed subset of \( Y \) and \( U \) be any \( \pi \)-open subset of \( X \) such that \( f^{-1}(A) \subset U \). Then by the Proposition 3.9, we have \( f(U) \) is a \( \pi \)-open subset of \( Y \) such that \( A \subset f(U) \). Since \( A \) is \( \pi \beta \)-closed subset of \( Y \) and \( f(U) \) is \( \pi \)-open set in \( Y \), thus \( \beta cl(A) \subset f(U) \). By the Lemma 3.7, we obtain that \( f^{-1}(A) \subset f^{-1}(\beta cl(A)) \subset \beta cl(A) \). Therefore, \( f^{-1}(A) \) is \( \pi \beta \)-closed set in \( X \).

3.11 Theorem. \( \pi \gamma \beta \)-normality is a topological property.

Proof. Let \( X \) be a \( \pi \gamma \beta \)-normal space and \( f : X \to Y \) be an open and closed bijection continuous function. We need to show that \( Y \) is \( \pi \gamma \beta \)-normal. Let \( A \) and \( B \) be any disjoint \( \pi \gamma \beta \)-closed subsets of \( X \). Then by the Proposition 3.10, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( \pi \gamma \beta \)-closed subsets of \( X \). By \( \pi \gamma \beta \)-normality of \( X \), there exist \( \beta \)-open subsets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). Since \( U \cap V = \emptyset \), we have \( A \subset f(U) \) and \( B \subset f(V) \). Hence, \( Y \) is \( \pi \gamma \beta \)-normal.

4. \( \pi \gamma \beta \)-normality in subspaces

4.1 Lemma. If \( M \) be an open subspace of a space \( X \) and \( A \subset M \), then \( \beta cl_{\beta}(A) = \beta cl_{X}(A) \cap M \).

4.2 Lemma [12]. If \( M \) be an open subspace of a space \( X \) and \( A \subset M \), then \( int_{\beta}(cl_{\beta}(A)) = int_{X}(cl_{X}(A)) \cap M \).

4.3 Lemma [12]. If \( M \) be a \( \pi \)-open subspace of a space \( X \) and \( U \) be a \( \pi \)-open subset of \( X \), then \( U \cap M \) is \( \pi \)-open set in \( M \).

4.4 Lemma. If \( A \) be both \( \pi \)-open and \( \pi \gamma \beta \)-closed subset of a space \( X \), then \( A \) is \( \beta \)-closed set in \( X \).

4.5 Corollary. If \( A \) is both \( \pi \)-open and \( \pi \gamma \beta \)-closed subset of a space \( X \), then \( A \) is regular closed set in \( X \).

4.6 Theorem. Let \( M \) be a \( \pi \)-open subspace of a space \( X \) and \( A \subset M \). If \( M \) be \( \pi \gamma \beta \)-closed set in \( X \) and \( A \) is \( \pi \gamma \beta \)-closed set in \( M \). Then \( A \) is \( \pi \gamma \beta \)-closed set in \( X \).

Proof. Suppose that \( M \) is \( \pi \gamma \beta \)-closed set in \( X \) and \( A \) is \( \pi \gamma \beta \)-closed set in \( M \). Let \( U \) be any \( \pi \)-open subset in \( X \) such that \( A \subset U \). Then by Lemma 4.3, we have \( A \subset M \cap U \), where \( M \cap U \) is \( \pi \)-open set in \( M \). Since \( A \) is \( \pi \gamma \beta \)-closed in \( M \), thus \( \beta cl_{M}(A) \subset M \cap U \). By the Lemma 4.4, we obtain that \( \beta cl_{X}(M) = M \). Hence, \( \beta cl_{X}(M) \subset U \). Therefore, \( A \) is \( \pi \gamma \beta \)-closed set in \( X \).

4.7 Lemma. Let \( M \) be a closed domain subspace of a space \( X \). If \( U \) be \( \beta \)-open set in \( X \), then \( U \cap M \) be \( \beta \)-open set in \( M \).

4.8 Theorem. \( \pi \gamma \beta \)-closed and \( \pi \)-open subspace of a space \( X \) and \( A \subset M \).

Proof. Let \( M \) be a \( \pi \gamma \beta \)-closed and \( \pi \)-open subspace of a space \( X \). Let \( A \) and \( B \) be any disjoint \( \pi \gamma \beta \)-closed subsets of \( M \). Then by Theorem 4.6, we have \( A \) and \( B \) be disjoint \( \pi \gamma \beta \)-closed sets in \( X \). By \( \pi \gamma \beta \)-normality of \( X \), there exist \( \beta \)-open subsets \( U \) and \( V \) of \( X \) such that \( A \subset U \), \( B \subset V \), \( U \cap V = \emptyset \). By the Corollary 4.5 and Lemma 4.7, we obtain that \( U \cap M \) and \( V \cap M \) are disjoint \( \beta \)-open sets in \( M \) such that \( A \subset U \cap M \) and \( B \subset V \cap M \). Hence, \( M \) is \( \pi \gamma \beta \)-normal space of \( \beta \)-normal space \( X \).

5. Preservation theorems for \( \pi \gamma \beta \)-Normality

5.1 Definition

A function \( f : X \to Y \) is said to be \( \pi \)-irresolute [3] if \( f^{-1}(F) \) is \( \pi \)-closed in \( X \) for every \( \pi \)-closed set \( F \) in \( Y \).

5.2 Theorem

If \( f : X \to Y \) is \( \pi \)-irresolute, \( \beta \)-closed and \( A \) is a \( \pi \gamma \beta \)-closed subset of \( X \), then \( f(A) \) is \( \pi \gamma \beta \)-closed subset of \( Y \).

Proof. Let \( A \) be a \( \pi \gamma \beta \)-closed subset of \( X \) and \( U \) be any \( \pi \)-open set of \( Y \) such that \( f(A) \subset U \). Then, \( A \subset f^{-1}(U) \). Since \( f \) is \( \pi \)-irresolute function, then \( f^{-1}(U) \) is \( \pi \)-open in \( X \). Since \( A \) is \( \pi \gamma \beta \)-closed set in \( X \) and \( A \subset f^{-1}(U) \), then \( \beta cl_{X}(A) \subset f^{-1}(U) \). This implies that \( f(\beta cl_{X}(A)) \subset U \). Since \( f \) is \( \beta \)-closed and \( \beta cl_{X}(A) \) is \( \beta \)-closed set in \( Y \), then \( f(\beta cl_{X}(A)) \) is \( \beta \)-closed set in \( Y \). Thus, we have \( \beta cl_{Y}(f(A)) \subset U \). Hence, \( f(A) \) is \( \pi \gamma \beta \)-closed subset of \( Y \).
5.3 Corollary

If $f : X \to Y$ is $\pi$-continuous, pre $\beta$-closed and $A$ is a $\pi\beta$-closed subset of $X$, then $f(A)$ is $\pi\beta$-closed subset of $Y$.

5.4 Theorem

If $f : X \to Y$ is $\pi$-irresolute, pre $\beta$-closed and $\beta$-irresolute injection function from a space $X$ to a $\pi\beta$-normal $Y$, then $X$ is $\pi\beta$-normal.

Proof. Let $A$ and $B$ be any two disjoint $\pi\beta$-closed subsets of $X$. By the Theorem 5.2 $f(A)$ and $f(B)$ are disjoint $\pi\beta$-closed subsets of $Y$. By $\pi\beta$-normality of $Y$, there exist disjoint $\beta$-open subsets $U$ and $V$ of $Y$ such that $f(A) \subset U$, $f(B) \subset V$ and $U \cap V = \emptyset$. Since $f$ is $\beta$-irresolute injection function, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $\beta$-open sets in $X$ such that $A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$. Hence $X$ is $\pi\beta$-normal.

5.5 Corollary

If $f : X \to Y$ is $\pi$-continuous, pre $\beta$-closed and $\beta$-irresolute injection function from a space $X$ to a $\pi\beta$-normal $Y$, then $X$ is $\pi\beta$-normal.

5.6 Lemma

If the bijection function $f : X \to Y$ is $\beta$-continuous and regular open, then $f$ is $\pi\beta$-irresolute.

5.7 Theorem

If $f : X \to Y$ is $\pi\beta$-irresolute, pre $\beta$-closed bijection function from a $\pi\beta$-normal space $X$ to a space $Y$, then $Y$ is $\pi\beta$-normal.

Proof. Let $A$ and $B$ be any two disjoint $\pi\beta$-closed subsets of $Y$. Since $f$ is $\pi\beta$-irresolute, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\pi\beta$-closed subsets of $X$. By $\pi\beta$-normality of $X$, there exist disjoint $\beta$-open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \emptyset$. Since $f$ is pre $\beta$-open and bijection function, we have $f(U)$ and $f(V)$ are disjoint $\beta$-open sets in $Y$ such that $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \emptyset$. Therefore, $X$ is $\pi\beta$-normal.

5.8 Corollary

If $f : X \to Y$ is $\beta$-continuous, regular open and pre $\beta$-open bijection function from a $\pi\beta$-normal space $X$ to a space $Y$, then $Y$ is $\pi\beta$-normal.

5.9 Theorem

If $f : X \to Y$ is a pre $\beta$-open, $\pi\beta$-irresolute and almost $\beta$-irresolute surjection function from a $\pi\beta$-normal space $X$ onto a space $Y$, then $Y$ is $\pi\beta$-normal.

References


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Umesh Chand received the M.Phil (Some new mappings in Topological spaces) from Chaudhary Charan Singh University, Meerut, UP in 2011