Smarandache Fuzzy Strong Ideal and Smarandache Fuzzy n-Fold Strong Ideal of a BH-Algebra

Shahrezad Jasim Mohammed

Abstract: In this paper, we define the concepts of a Q-Smarandache n-fold strong ideal and Q-Smarandache fuzzy (strong, n-fold strong) ideal of a BH-algebra. Also, we study some properties of these fuzzy ideals.

Keywords: BCK-algebra, BCI/BCH-algebras, BH-algebra, Smarandache BH-algebra, Q-Smarandache fuzzystrong ideal.

1. Introduction


2. Preliminaries

In this section, we give some basic concept about a BCK-algebra, A BCI-algebra, a BH-algebra, a BH*-algebra, a normal BH-algebra, fuzzy strong ideal, fuzzyn-fold strong ideal, Smarandache BH-algebra, (Q-Smarandache ideal, Q-Smarandache fuzzy closed ideal, Q-Smarandache fuzzy completely closed ideal) and Q-Smarandache fuzzy completely closed ideal of a BH-algebra.

Definition 1 (see[11]). A BCI-algebra is an algebra (X,*,0), where X is a nonempty set, "*" is a binary operation and 0 is a constant, satisfying the following axioms: for all x, y, z \in X:

i. ((x*y)*(x*z))=(z*(x*y))=0, ii. (x*(x*y))*y=0, iii. x*y=0, iv. x*y=0 = 0 \Rightarrow x = y.

Definition 2 (see [8]). A BCK-algebra is a BCI-algebra satisfying the axiom

v. 0 * x = 0 for all x \in X.

Definition 3 (see[9]). A BH-algebra is a nonempty set X with a constant 0 and a binary operation "*" satisfying the following conditions: i. x*x=0, \forall x \in X ii. x*0 =x, \forall x \in X. iii. x*y=0 and y*x =0 \Rightarrow x = y; for all x, y \in X.

Definition 4 (see[4]). A BH-algebra X is called a BH*-algebra if it satisfies the condition: i. x*(y*x)=0, \forall x \in X. ii. (x*y)*x = 0* y, \forall x, y \in X. iii. (x*(x*y))*y = 0, \forall x, y \in X.

Definition 5 (see[9]). A BH-algebra X is said to be a normalBH-algebra if it satisfies the following condition: i. 0*(x*y)= (0*x)*(0*y), \forall x, y \in X. ii. (x*y)x = 0 * y, \forall x, y \in X. iii. (x*(x*y))*y = 0, \forall x, y \in X.

Definition 6 (see[5]). Let X be a BH-algebra. Then the set X*=\{x \in X|0*x=0\} is called the BCA-part of X.

Remark 1 (see[6]). Let X and Y be BH-algebras. A mapping f: X \rightarrow Y is called a homomorphism if f(x*y)=f(x)*f(y) for all x, y \in X. A homomorphism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). For any homomorphism f:X \rightarrow Y, the set \{ x \in X | f(x)=0\} is called the kernel of f, denoted by Ker(f), and the set \{ f(x) : x \in X \} is called the image of f, denoted by Im(f). Notice that f(0)=0 for all homomorphism f.

Definition 7 (see[3]). Let X be a BH-algebra and n be a positive integer. A nonempty subset I of X is called a n-fold strong ideal of X if it satisfies: i. 0 \in I, ii. \forall y \in I and (x*y)*z=0 \in I \Rightarrow x*0 \in I, \forall x, z \in X.

Definition 8 (see[6]). A Smarandache BH-algebra is defined to be a BH-algebra X in which there exists a proper subset Q of X such that i. 0 \in Q \text{ and } |Q| \geq 2 , ii. Q is a BCK-algebra under the operation of X.

Definition 9 (see[6]). Let X be a Smarandache BH-algebra. A nonempty subset I of X is called a Smarandache strong ideal of X related to Q (or briefly, Q-Smarandache strong ideal of X) if it satisfies: i. 0 \in I , ii. \forall y \in I \text{ and } (x*y)*z \in I \Rightarrow x*0 \in I, \forall x, z \in Q.
Definition 10 (see [7]). Let \( \mu \) be a fuzzy set in \( X \), for all \( t \in [0,1] \). The set \( \mu = \{ x \in X, \mu(x) \geq t \} \) is called a level subset of \( \mu \).

**Definition 11 (see [4]).** Let \( A \) and \( B \) be any two sets, \( \mu \) be any fuzzy set in \( A \) and \( f : A \to B \) be any function. Set \( f^{-1}(y) = \{ x \in A | f(x) = y \} \) for \( y \in B \). The fuzzy set \( \nu \) in \( B \) defined by \( \nu(y) = \bigcap_{x \in f^{-1}(y)} \mu(x) \) is called the image of \( \mu \) under \( f \) and is denoted by \( f(\mu) \).

**Definition 12 (see [4]).** Let \( A \) and \( B \) be any two sets, \( f : A \to B \) be any function and \( \nu \) be any fuzzy set in \( f(A) \). The fuzzy set \( \mu \) in \( A \) defined by: \( \mu(x) = \nu(f(x)) \) for all \( x \in X \) is called the preimage of \( \nu \) under \( f \) and is denoted by \( f^{-1}(\nu) \).

**Definition 13 (see [4]).** A fuzzy set \( \mu \) in a BH-algebra \( X \) is called a fuzzy strong ideal of \( X \) iff i. For all \( x \in X \), \( \mu(0) \geq \mu(x) \).

**Definition 14 (see [5]).** A fuzzy set \( \mu \) in a BH-algebra \( X \) is called a fuzzy \( n \)-fold strong ideal of \( X \) iff i. For all \( x \in X \), \( \mu(0) \geq \mu(x) \).

**Definition 15 (see [6]).** A fuzzy subset \( \mu \) of a Smarandache BH-algebra \( X \) is said to be \( Q\)-Smarandache fuzzy ideal iff

1. for all \( x \in X \), \( \mu(0) \geq \mu(x) \).
2. for all \( x \in X \), \( \mu(x) \geq \min\{\mu((x*y)*z), \mu(y)\} \).

**Definition 16 (see [6]).** Let \( X \) be a Smarandache BH-algebra and \( \mu \) be a \( Q\)-Smarandache fuzzy ideal of \( X \). This \( \mu \) is called a \( Q\)-Smarandache fuzzy ideally closed ideal if \( \mu((x*y)*z) \geq \min\{\mu((x*y)*z), \mu(y)\} \) for all \( x \in X \).

**Proposition 1 (see [6]).** Let \( X \) be a BH-algebra and \( \mu \) be a \( Q\)-Smarandache fuzzy ideal. Then i. \( \mu \) is a \( Q\)-Smarandache fuzzy closed ideal of \( X \).

**Proposition 2 (see [6]).** Every fuzzy strong ideal of a Smarandache BH-algebra \( X \) is a \( Q\)-Smarandache fuzzy ideal of \( X \).

**The Main Results**

In this paper, we give the concepts a \( Q\)-Smarandache \( n \)-fold strong ideal and a \( Q\)-Smarandache fuzzy (strong, \( n \)-fold strong) ideal of a BH-algebra. Also, we give some properties of these fuzzy ideals.

**Definition 1.** A fuzzy subset of a BH-algebra \( X \) is called a \( Q\)-Smarandache fuzzy strong ideal if i. \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

**Example 1:** The set \( X = \{0, 1, 2, 3\} \) with the following operation table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

is a BH-algebra \( Q = [0,1] \) is a BCK-algebra. Then \( (X, *, 0) \) is a Smarandache BH-algebra. The fuzzy set \( \mu \) which is defined by:

\[
\mu(x) = \begin{cases} 0.5 & x = 0.3 \\ 0.4 & x = 1 \end{cases}
\]

is a \( Q\)-Smarandache fuzzy strong ideal, since: i. \( \mu(0) = 0.5 \), ii. \( \mu(x) \geq \min\{\mu((x*y)*z), \mu(y)\} \) for all \( x \in X \).

But the fuzzy set \( \mu \) is not a \( Q\)-Smarandache fuzzy strong ideal since \( \mu(1) = 0.4 < \min\{\mu((1*3)*0), \mu(3)\} = 0.5 \) .

**Proposition 1.** Every \( Q\)-Smarandache fuzzy strong ideal of a Smarandache BH-algebra \( X \) is a \( Q\)-Smarandache fuzzy ideal of \( X \).

**Proof:** Let \( \mu \) be a fuzzy strong ideal of \( X \).

1. Let \( x \in X \Rightarrow 0 \leq \mu(x) \).

2. Let \( x, z \in X \) and \( y \in X \Rightarrow (x, z) \in Q \)

\[
\Rightarrow \mu((x*y)*z) \geq \min\{\mu((x*y)*z), \mu(y)\}
\]

**Proposition 2.** Let \( Q_1 \) and \( Q_2 \) be a BCK-algebra contained in a Smarandache BH-algebra \( X \) and \( Q_1 \subseteq Q_2 \). Let \( \mu \) be a \( Q\)-Smarandache fuzzy strong ideal of \( X \) then \( \mu \) is a \( Q_1\)-Smarandache fuzzy strong ideal of \( X \).

**Proof:** Let \( \mu \) be a \( Q_2\)-Smarandache fuzzy strong ideal of \( X \).

1. Let \( x \in X \Rightarrow 0 \leq \mu(x) \).

2. Since \( \mu \) is a \( Q_1\)-Smarandache fuzzy strong ideal by definition 1(i)

\[
\Rightarrow \mu((x*y)*z) \geq \min\{\mu((x*y)*z), \mu(y)\}
\]

**Proposition 3.** Every fuzzy strong ideal of a Smarandache BH-algebra \( X \) is a \( Q\)-Smarandache fuzzy strong ideal of \( X \).

**Proof:** Let \( \mu \) be a fuzzy strong ideal of \( X \).

1. Let \( x \in X \Rightarrow 0 \leq \mu(x) \).

2. Since \( \mu \) is a \( Q_1\)-Smarandache fuzzy strong ideal by definition 1(i)

\[
\Rightarrow \mu((x*y)*z) \geq \min\{\mu((x*y)*z), \mu(y)\}
\]

**Theorem 1.** Let \( X \) be Smarandache BH-algebra and let \( \mu \) be a fuzzy set. Then \( \mu \) is a \( Q\)-Smarandache fuzzy strong ideal if and only if \( \nu(x) = \mu(x) / \mu(0) \) is a \( Q\)-Smarandache fuzzy strong ideal.

**Proof:** Let \( \mu \) be a \( Q\)-Smarandache fuzzy strong ideal, then:

1. \( \nu(0) = \mu(0) / \mu(0) \Rightarrow \nu(0) = 1 \Rightarrow \nu(x) \geq \nu(x) \) for all \( x \in X \).
Theorem 2. Let X be a non-empty subset of a Q-Smarandache BH-algebra X and let μ be a fuzzy set in X defined by:

$$μ(x) = \begin{cases} α₁ & \text{if } x \in EQ \\ α₂ & \text{otherwise} \end{cases}$$

where $α₁ > α₂ \geq 0$. Then μ is a Q-Smarandache fuzzy strong ideal of X.

Proof: Let μ be a fuzzy set of X.

i. If x ∈ EQ, then μ(x) = α₁ ≥ μ(y) \[By definition 1(ii)\]

ii. If x ∈ EQ and y ∈ X then x * y ∈ EQ.

Then we have for cases.

Case 1: If $(x*y) \neq 0$ and $μ(y) = α₁$ then:

$$max(μ((x*y)*z), μ(y)) = α₁ ≥ α₁$$

$$μ((x*y)*z) = μ(x*y) + 1$$

Case 2: If $(x*y) \neq 0$ and $μ(y) = α₂$ then:

$$min(μ((x*y)*z), μ(y)) = α₁ ≥ α₁$$

$$μ((x*y)*z) = μ(x*y) + 1$$

Case 3: If $(x*y) \neq 0$ and $μ(y) = α₂$ then:

$$min(μ((x*y)*z), μ(y)) = α₂ ≥ α₂$$

$$μ((x*y)*z) = μ(x*y) + 1$$

Then μ is a Q-Smarandache fuzzy strong ideal of X.

Theorem 3. Let X be BH-algebra and let μ be a fuzzy set. Then μ is a Q-Smarandache fuzzy strong ideal if and only if μ(x) = μ(x)+1 - μ(0) is a Q-Smarandache fuzzy strong ideal.

Proof: Let μ be a Q-Smarandache fuzzy strong ideal, i. $μ(0) = μ(0)+1 - μ(0)$, $μ(x) = μ(x)+1 - μ(x) \forall x \in X$

ii. $μ(x*y) = μ((x*y)*z)$, $μ(y) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$

Then we have for cases.

Conversely, let X be a fuzzy set of X, and μ be a Q-Smarandache fuzzy strong ideal. Then μ is a Q-Smarandache fuzzy strong ideal if and only if μ(x) = μ(x)+1 - μ(0) is a Q-Smarandache fuzzy strong ideal.

Proof: Let μ be a Q-Smarandache fuzzy strong ideal, i. $μ(0) = μ(0)+1 - μ(0)$, $μ(x) = μ(x)+1 - μ(x) \forall x \in X$

ii. $μ(x*y) = μ((x*y)*z)$, $μ(y) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$

Then we have for cases.

Theorem 4. Let X be a BH-algebra and μ be a Q-Smarandache fuzzy strong ideal. Then μ(x) = μ(x)+1 - μ(x) and μ(0) = μ(0)+1 - μ(0) are Q-Smarandache fuzzy strong ideals.

Proof: Let μ be a Q-Smarandache fuzzy strong ideal, i. $μ(0) = μ(0)+1 - μ(0)$, $μ(x) = μ(x)+1 - μ(x) \forall x \in X$

ii. $μ(x*y) = μ((x*y)*z)$, $μ(y) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$, $μ(0) = μ((x*y)*z)$

Then we have for cases.

Conversely, let μ be a Q-Smarandache fuzzy strong ideal. Then μ(x) = μ(x)+1 - μ(x) and μ(0) = μ(0)+1 - μ(0) are Q-Smarandache fuzzy strong ideals.
\( \Rightarrow \mu \) is a Q-Smarandache fuzzy ideal of \( X \). [By proposition 1]

Now, let \( x, y \in X \). \( x^y \in Q \) since \( X^X \times \{0\} \subseteq Q \)

\( \Rightarrow \mu(x^y) \geq \min \{ \mu((x*y)x^y) \}, \mu(x) \} \) [By definition 1]

\( = \min \{ \mu(0^x y^z), \mu(x) \} \) [By definition 5(ii)]

\( \geq \min \{ \mu(y), \mu(x) \} \) [By definition 14]

\( \Rightarrow \mu \) is a Q-Smarandache fuzzy completely closed ideal of \( X \).

Remark 1. Let \( \mu \) be a fuzzy set of a Smarandache BH-algebra \( X \) and \( w \in X \). The set \( \{ x \in X : (w) \leq \mu(x) \} \) is denoted by \( \mathcal{T}(w) \).

Theorem 5. Let \( X \) be a Smarandache BH-algebra, \( w \in X \) and \( \mu \) is a Q-Smarandache fuzzy strongly ideal. Then \( \mathcal{T}(w) \) is a Q-Smarandache strongly ideal.

Proof: Let \( \mu \) be a Q-Smarandache fuzzy strongly ideal of \( X \). To prove that \( \mathcal{T}(w) \) is a Q-Smarandache strongly ideal of \( X \).

1. Let \( x \in \mathcal{T}(w) \). Then \( \mu(x) \geq \mu(w) \) [By definition 1(i)]

\( \Rightarrow (0) \geq \mu(w) \Rightarrow 0 \in \mathcal{T}(w) \)

2. Let \( x \in Q \), \( y \in \mathcal{T}(w) \) and \( (x*y)z \in \mathcal{T}(w) \).

\( \Rightarrow (w) \leq \mu(y) \) and \( (w) \leq \mu((x*y)z) \)

\( \Rightarrow (w) \leq \mu(y) \) and \( (w) \leq \mu((x*y)z) \)

But \( \mu(x^y) \geq \min \{ \mu(x^y), \mu(y) \} \) [By definition 1(ii)]

\( \Rightarrow (w) \leq \mu(x^y) \Rightarrow x^y \in \mathcal{T}(w) \)

\( \Rightarrow \mathcal{T}(w) \) is a Q-Smarandache strongly ideal of \( X \).

Corollary 1. Let \( X \) be a Smarandache BH-algebra. Then \( \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \) if and only if \( \mu(x) \) is a Q-Smarandache strongly ideal of \( X \), for all \( t \in [0, \sup \mu(x)] \).

Proof: Let \( t \in [0, \sup \mu(x)] \). To prove that \( \mu(x) \) is a Q-Smarandache fuzzy strongly ideal of \( X \). Since \( \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Now, let \( y \in \mu(x) \) and \( x^y \subseteq \mu(x) \Rightarrow (y) \geq t \) and \( (x*y)z \subseteq \mu(x) \) \( t \geq \).

To prove that \( (x*y)z \subseteq \mu(x) \).

We have \( \mu(x^y) \geq \min \{ \mu((x*y)z), \mu(y) \} \) [By definition 1]

Since \( \mu(x^y) \geq t \) and \( \mu(y) \geq t \Rightarrow \min \{ \mu((x^y)z), \mu(y) \} \geq t \)

\( \Rightarrow \mu(z) \geq t \Rightarrow x^y \in \mu \)

\( \Rightarrow \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Conversely, To prove that \( \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Since \( \mu \) is a Q-Smarandache strongly ideal of \( X \).

Let \( t = \sup \mu(x), x \in \mathcal{E}(\mathcal{X}) \) and \( (x*y)z, y \in \mu \)

\( \Rightarrow x^y \subseteq \mu(x) \) [By definition 9]

\( \Rightarrow \mu(x^y) \geq t \Rightarrow \mu(x^y) = t \) [Since \( t = \sup \mu(x) \)]

Similarly, \( \mu(x^y) \geq t \) and \( \mu(y) = t \)

\( \Rightarrow t \) \( = \min \{ \mu(x^y), \mu(y) \} \)

\( \Rightarrow \mu(x^y) \geq \min \{ \mu(x^y), \mu(y) \} \)

\( \Rightarrow \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Proposition 7. Let \( f : (X, *, 0) \rightarrow (Y, \ast', 0') \) be a Smarandache BH-epimorphism. If \( \mu \) is a Q-Smarandache fuzzy strongly ideal of \( X \), then \( f(\mu) \) is a \( f(Q) \)-Smarandache fuzzy strongly ideal of \( Y \).

Proof: Let \( \mu \) be a Q-Smarandache fuzzy strongly ideal of \( X \). i. Let \( y \in f(\mu) \) such that \( x = f(x) \).

\( (f(\mu))^{(0')} = \sup \{ \mu(x) | x \in f^{-1}(0') \} \)

\( = (0) \geq \mu(x) \) [By definition 1(i)]

\( = (f(\mu))(f(x)) \)

\( = (f(\mu))(y) \Rightarrow (f(\mu))(y) \Rightarrow (f(\mu))(y) \)

\( \Rightarrow \mathcal{T}(\mu) \) \( \Rightarrow \min \{ (f(\mu))((y), y, y), (f(\mu))(y)) \) \( \Rightarrow \mathcal{T}(\mu) \) \( \Rightarrow \mathcal{T}(\mu) \) is a Q-Smarandache fuzzy strongly ideal of \( Y \).

Theorem 6. Let \( f : (X, *, 0) \rightarrow (Y, \ast', 0') \) be a Smarandache BH-epimorphism. If \( f \) is a Q-Smarandache fuzzy strongly ideal of \( X \), then \( f^{-1}(\mathcal{Q}) \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Proof: i. Let \( x \in X \). Since \( f(x) \in \mathcal{E}(\mathcal{Y}) \) and \( \mathcal{Q} \) is a Q-Smarandache strongly fuzzy ideal of \( Y \).

\( \Rightarrow f^{-1}(\mathcal{O}) = \mathcal{E}(f(x)) \) [By definition 12]

\( \Rightarrow \min \{ \mathcal{E}(f(x)), \mathcal{E}(y) \} \) [By remark 1]

\( \Rightarrow f^{-1}((\mathcal{E}(x) \ast y), (x \ast y), (y)) \] [By definition 11]

\( \Rightarrow f^{-1}(\mathcal{Q}) \) is a Q-Smarandache fuzzy strongly ideal of \( X \).

Definition 2. Let \( X \) be a Smarandache BH-algebra and \( \mu \) be a positive ideal. A nonempty subset I of \( X \) is called a Smarandachen-fold strong ideal of \( X \) related to \( Q \) (or briefly, Q-Smarandachen-fold strong ideal of \( X \) if it satisfies: i. \( 0 \in I \), \( x \in I \) and \( (x*y)z \in I \) \( \Rightarrow x^y \in I \), \( \forall x, z \in Q \).

Example 2. Consider the set \( I = \{0,3\} \) in example 1 is a Q-Smarandachen-fold strong ideal of \( X \). But the set \( I = \{0,2,3\} \) is not a Q-Smarandachen-fold strong ideal since \( 1 \ast 0 \in I \) and \( 1 \ast 3 \in I \).

Remark 2. Every n-fold strong ideal of a Smarandache BH-algebra \( X \) is a Q-Smarandachen-fold strong ideal of \( X \).

Remark 3. Every Q-Smarandachen-fold strong ideal of a Smarandache BH-algebra \( X \) is a Q-Smarandachen-fold strongly ideal of \( X \).

Remark 4. Every Q-Smarandachen-fold strongly ideal of a Smarandache BH-algebra \( X \) is a Q-Smarandache strongly ideal of \( X \).

Proposition 8. Let \( X \) be a Smarandache BH-algebra. Then every Q-Smarandachen-fold strongly ideal of \( X \) is contained in \( Q \) as a Q-Smarandache completely closed ideal of \( X \).

Proof: Let \( I \) be a Q-Smarandachen-fold strongly ideal of \( X \) \( \Rightarrow I \) is a Q-Smarandache globally ideal of \( X \). [By remark 4]

Now, let \( x, y \in I \) \( \Rightarrow x, y \in \mathcal{E}(Q) \) \( \Rightarrow x^y \in Q \) [Since \( I \subseteq Q \).]
Where \(( x*y )^n * x \overset{=}=( x*y )^n \). By definition 1(iii)
\[ = (0*y)^n \overset{[\text{By definition 3(iii)}]}{=} 0^y \overset{[\text{By definition 3(ii)}]}{=} \varepsilon \varepsilon \left[ \text{Since} \ 0*x = 0. \text{By definition 2(v)} \right]
\[ \Rightarrow ((x*y)^n*0)^\varepsilon \in X \Rightarrow \varepsilon y \in \varepsilon \in [\text{By definition 3(iii)}]
\Rightarrow I \text{ is a Q-Smarandache completely closed ideal of X.}
\]

**Definition 3.** A fuzzy subset of a Smarandache BH-algebra X and n be a positive integer is called a **Q-Smarandache fuzzy n-fold strong ideal** if i. \( \mu(0) \geq \mu(x)*y \) \( \forall x \in X \). ii. \( \mu(x*z) \geq \min(\mu((x*y)^n)\mu(y)) \), \( \forall x, y \in X \).

**Example 3.** Consider the fuzzy set \( \mu \) which is defined by:
\[ \mu(x) = \begin{cases} 0.2, & x = 0, \\ 1.3, & x = 1, \\ 0.4, & x = 1, \end{cases} \]
\[ \text{is a Q-Smarandache fuzzy n-foldstrong ideal, since:} \ i. \mu(0) = 0.5 \mu(x) \forall x \in X. \ ii. \mu(x*z) \geq \min(\mu((x*y)\mu(y))) \forall x, y \in X. \]

But the fuzzy set \( \mu(x) = \begin{cases} 0.2, & x = 0, \\ 1.3, & x = 1, \\ 0.4, & x = 1, \end{cases} \]

is not a Q-Smarandache fuzzy n-foldstrong ideal since \( \mu(1*0) = \mu(1) = 0.4 < \min(\mu(1*3)*0), \mu(3)) = 0.5 \)

**Remark 5.** Every fuzzy n-fold strong ideal of a Smarandache BH-algebra X is a Q-Smarandache fuzzy n-fold strong ideal of X.

**Proposition 9.** Every Q-Smarandache fuzzy n-fold strong ideal of a Smarandache BH-algebra X is a Q-Smarandache fuzzy n-fold strong ideal of X.

**Proof:** Let \( \mu \) be a Q-Smarandache fuzzy n-fold strong ideal of X. i. Let \( x \in X \Rightarrow (0) \geq \mu(x) \) \( \text{[By definition 3(iii)]} \)

ii. Let \( x, y \in X \Rightarrow \mu(x*z) \geq \min(\mu((x*y)^n)\mu(y)) \), \( \forall x, y \in X \).

Then \( \bigcap \mu_a \) is a fuzzy n-fold strong ideal of X.

**Proposition 10.** Let \( \{\mu_a: a \in \Lambda\} \) be a family of Q-Smarandache fuzzy n-fold strong ideals of X. Then:

**Proof:** Let \( \{\mu_a: a \in \Lambda\} \) be a family of Q-Smarandache fuzzy n-fold strong ideals of X. Then:

i. Let \( x \in X \Rightarrow \mu_a(0) = \inf(\mu_a(0), a \in \Lambda) \geq \inf(\mu_a(x), a \in \Lambda) \)

ii. Let \( x, y \in X \Rightarrow \mu(x*z) \geq \min(\inf(\mu_a(x*y)^n, a \in \Lambda), \inf(\mu_a(y), a \in \Lambda)) \)

\[ \Rightarrow \bigcap \mu_a \] is a Q-Smarandache fuzzy n-fold strong ideal of X.

**Theorem 7.** Let \( f : (X,*,0) \rightarrow (Y,*,0') \) be a Smarandache BH-epimorphism. If \( X \) is a Q-Smarandache fuzzy n-fold strong ideal of Y, then \( f^{-1}(X) \) is a Q-Smarandache fuzzy n-fold strong ideal of X.

**Proof:** i. Let \( x \in X \). Since \( f(x) \in Y \) and \( y \) is a Q-Smarandache fuzzy n-fold strong ideal of Y.

\( f^{-1}(y)(0) = (0) \Rightarrow f(0') \geq v(f(x)) = (f^{-1}(v))(x) \)

ii. Let \( x, y \in X \Rightarrow f^{-1}(Q), y \in X \)

\( f^{-1}(x*y) = (f^{-1}(x*y)) \) \[ \text{[By definition 11]} \]

\[ \geq \min(\mu(f(x)*f(y)*f(z)), v(f(y))) \]

[By remark1]

\[ = \min(\mu(f((x*y)*z)), v(f(y))) \]

\[ \Rightarrow f^{-1}(v)(x*y) \geq \min(\mu((x*y)*z)), f^{-1}(v)(y)) \]

\[ \Rightarrow f^{-1}(v) \text{ is a Q-Smarandache fuzzy n-fold strong ideal of X.} \]

**Theorem 8.** Let X be a Smarandache BH-algebra. If \( \mu \) is a fuzzy set such that Q=

\[ \mu \in \{ \mu(x) \in \mu(0) \} \text{ and } \mu(0) \geq \mu(x) \forall x \in X, \text{ then } \mu \text{ is a Q-Smarandache fuzzy n-fold strong ideal of X.} \]

**Proof:** Let \( \mu \) be a fuzzy set of X, such that Q = X and \( \mu(0) \geq \mu(x) \forall x \in X. \)

i. \( 0 \geq \mu(x) \forall x \in X. \)

ii. Let \( x, y \in X \text{ and } y \in X \),

\[ \mu(0) = \inf(\mu(0), a \in \Lambda) \geq \inf(\mu(x), a \in \Lambda) \]

[Since \( \mu_a \text{ is a Q-Smarandache fuzzy n-fold ideal, } \forall a \in \Lambda. \text{ By definition 3(i)}]
\[
\bigcup_{\alpha \in \lambda} \mu_\alpha(x) \Rightarrow \bigcup_{\alpha \in \lambda} \mu_\alpha(0) \geq \bigcup_{\alpha \in \lambda} \mu_\alpha(x) \forall x \in X.
\]

ii. Let \( x, z \in Q \), \( y \in X \).

\[
\mu_\alpha(x^nz^n) = \sup\{ \mu_\alpha(x), \alpha \in \lambda \} \geq \sup\{ \min\{ \mu_\alpha((xy)^nz^n), \mu_\alpha(y) \}, \alpha \in \lambda \}
\]

[Since \( \mu_\alpha \) is a Q-Smarandache fuzzy n-fold strong ideal, \( \forall \alpha \in \lambda \). By definition 3(iii)]

But \( \{ \alpha, \alpha \in \lambda \} \) is a chain \( \Rightarrow \) there exist, \( j \in \lambda \) such that

\[
\mu_\alpha(x^nz^n) \geq \min\{ \mu_\alpha((xy)^nz^n), \mu_\alpha(y) \} \geq \min\{ \sup\{ \mu_\alpha((xy)^nz^n), \alpha \in \lambda \}, \sup\{ \mu_\alpha(y), \alpha \in \lambda \} \}
\]

[Since \( \mu_\alpha \) is a Q-Smarandache fuzzy n-fold strong ideal of \( X \).]

\[
\bigcup_{\alpha \in \lambda} \mu_\alpha(x^nz^n) \geq \min\{ \sup\{ \mu_\alpha((xy)^nz^n), \alpha \in \lambda \}, \sup\{ \mu_\alpha(y), \alpha \in \lambda \} \}
\]

\[
\bigcup_{\alpha \in \lambda} \mu_\alpha(x^nz^n) \geq \min\{ \bigcup_{\alpha \in \lambda} \mu_\alpha((xy)^nz^n), \bigcup_{\alpha \in \lambda} \mu_\alpha(y) \}
\]

References


