

Stability of Triangular Libration Points in the Perturbed Photogravitational Restricted Problem of Three Bodies

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Abstract: We investigate the linear stability of triangular libration points in the Perturbed Photogravitational Restricted Problem of Three Bodies in which both the primaries are source of radiation and small perturbations α and β are given to the Coriolis and centrifugal forces respectively. We observed that the triangular points (L_4, L_5) are stable under certain condition in terms of critical mass ratio.

Keywords: Photogravitational, Primary, pPRTBP, Stability, Critical Mass

1. Introduction

The restricted three-body problem is a simplified form of the general three-body problem, in which one of the body is of infinitesimal mass and other two are massive bodies (primaries). Two bodies revolve around their centre of mass in circular orbits under the effect of their mutual gravitational attraction and a third body (attracted by the previous two but not effecting their motion) moves in the plane defined by the two revolving bodies (primaries).

The Circular restricted three-body problem (CR3BP) becomes photogravitational restricted three-body problem when both primaries are radiating. The perturbed photogravitational restricted problem of three bodies possesses five equilibrium points, three collinear and two triangular. The collinear points L_1, L_2, L_3 lie on the line joining the primaries and are unstable while the triangular points L_4, L_5 form equilateral triangles with primaries and are stable if the mass parameter μ is less than the critical mass value $\mu_c = 0.0398521\dots$ (Szebehely, 1967).

Wintner (1941) Showed that the stability of the two triangular points is due to existence of the Coriolis terms in the equations of motion written in a synodic coordinates system. Szebehely (1967) considered the effect of small perturbation of the Coriolis force on the stability of the equilibrium points keeping the centrifugal force constant. He proved that the collinear points remain unstable and for the stability of the triangular points he obtained the relation

$$\mu_c = \mu_0 + \frac{16\epsilon}{3\sqrt{69}}$$

between the critical value of the mass parameter μ_c and the change ' ϵ ' in the Coriolis force. He established that the Coriolis force is a stabilizing force. Subbarao and Sharma (1975) examined the problem taking one of the primaries as an oblate spheroid whose equatorial plane coincides with the plane of motion. They proved that the range of linear stability of the triangular points decreases thereby establishing that the Coriolis force is not always a stabilizing force.

Bhatnagar and Hallan (1978) studied the effect of perturbations ϵ and ϵ' in Coriolis and centrifugal forces

respectively establishing that the collinear points remain unstable and the range of stability of the triangular points increases or decreases depending upon whether the points (ϵ, ϵ') lies in one or the other of the two parts in which the (ϵ, ϵ') plane is divided by the line $36\epsilon - 19\epsilon' = 0$.

Manju and Chaudhary (1985) studied the stability of the triangular equilibrium points taking into account the radiation pressure in the circular restricted problem. Vijay Kumar and Chaudhary (1987) investigated the stability of the triangular solution of the problem in which both the primaries are radiating ones under the non-resonance cases.

Singh and Umar (2012) has been investigated, and it is found that triangular points are stable if the eccentricity satisfies the condition

$0 \leq e \leq (\sqrt{7}/4)[1 - (9/7)A(1-\mu)]$ and the mass ratio obeys the inequality $0 < \mu < \mu_c$, where μ_c depends on the joint effects of the parameters involved.

Haque and Kumar (2014) has been found, that triangular equilibrium points are stable. Moreover the range of the stability of triangular equilibrium points decreases on account of oblateness and photogravitational effect of the primaries. When oblateness and radiating effects are ignored, the critical value reduces to $\mu_c = 0.0398521\dots$

In the present paper, we have studied the linear stability of triangular libration points in the perturbed photogravitational restricted problem of three bodies in which both the primaries are radiating ones after introducing small perturbations α and β in the Coriolis and centrifugal forces respectively. The force of radiation is given by the equation

$$F_p = qF_g$$

F_p being the radiation repulsive force and F_g being the gravitational attraction force; $q < 1$.

The equation of motion of the three body is derived in a synodic coordinate system using dimensionless variables. The canonical units remain same as in the ideal restricted problem. It is assumed that, due to the heavy mass of the

primaries, their radiation effect do not affect their mean motion.

Linear stability of the triangular libration points is investigated. It is found that the triangular points form nearly equilateral triangles with the primaries and stable if μ (mass parameter) is less than μ_c (critical mass). The range of linear stability depends upon the mass reduction factor, Coriolis force and centrifugal force. We observed that the stability increases with the increase of mass reduction factor and Coriolis force but stability decreases with the increase of centrifugal force.

2. Stability

We know that the equation of motion of perturbed photogravitational restricted problem of three bodies are given by

$$\begin{aligned} x - 2\alpha y &= \frac{\partial \Omega}{\partial x} \\ y + 2\alpha x &= \frac{\partial \Omega}{\partial y} \end{aligned} \quad \dots\dots(1)$$

$$\text{where } \Omega = \frac{\beta}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} \quad \dots\dots(2)$$

$$\begin{aligned} \text{and } r_1^2 &= (x + \mu)^2 + y^2 \\ r_2^2 &= (x - 1 + \mu)^2 + y^2 \end{aligned} \quad \dots\dots(3)$$

Where q_1, q_2 are mass reduction factors.

The equilibrium points are the solutions of the equations

$$\Omega_x = \Omega_y = 0 \quad \dots\dots\dots(4)$$

Solving the above equations we find the coordinates of the triangular points as

$$x = \frac{1}{2} - \mu + \frac{q_1^{2/3} - q_2^{2/3}}{2\beta^{2/3}} \quad \dots\dots(5)$$

$$y = \pm \left[\left(\frac{q_1^{2/3} + q_2^{2/3}}{2\beta^{2/3}} \right) - \left(\frac{q_1^{2/3} - q_2^{2/3}}{2\beta^{2/3}} \right)^2 - \frac{1}{4} \right]^{1/2} \quad \dots\dots(6)$$

Now, we examine the linear stability of the triangular libration points, for this we have

$$\begin{aligned} \Omega_{xx} &= \beta - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} + \frac{3(1-\mu)(x+\mu)^2 q_1}{r_1^5} \\ &\quad + \frac{3\mu(x-1+\mu)^2 q_2}{r_2^5} \end{aligned} \quad \dots\dots\dots(7)$$

$$\Omega_{xy} = \left[\frac{3(1-\mu)(x+\mu)q_1}{r_1^5} + \frac{3\mu(x-1+\mu)q_2}{r_2^5} \right] y \quad \dots\dots(8)$$

$$\begin{aligned} \Omega_{yy} &= \beta - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3} + \left[\frac{3(1-\mu)q_1}{r_1^5} \right. \\ &\quad \left. + \frac{3\mu q_2}{r_2^5} \right] y^2 \end{aligned} \quad \dots\dots(9)$$

Taking

$$\begin{aligned} \alpha &= 1 + \epsilon_1; \quad |\epsilon_1| \ll 1. \\ \beta &= 1 + \epsilon_2; \quad |\epsilon_2| \ll 1. \\ q_1 &= 1 - \delta_1; \quad |\delta_1| \ll 1. \\ q_2 &= 1 - \delta_2; \quad |\delta_2| \ll 1. \end{aligned} \quad \dots\dots\dots(10)$$

We get from

$$r_1^3 = \frac{q_1}{\beta} \quad \dots\dots(11)$$

$$r_1 = 1 - \frac{1}{3}\delta_1 - \frac{1}{3}\epsilon_2 \quad \dots\dots(12)$$

And

$$r_2^3 = \frac{q_2}{\beta} \quad \dots\dots(13)$$

$$r_2 = 1 - \frac{1}{3}\delta_2 - \frac{1}{3}\epsilon_2 \quad \dots\dots(14)$$

Also

$$x = \frac{1}{2} - \mu - \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2 \quad \dots\dots(15)$$

$$(x + \mu)^2 = \frac{1}{4} - \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2 \quad \dots\dots(16)$$

$$y^2 = \frac{3}{4} \left[1 - \frac{4}{9}\delta_1 - \frac{4}{9}\delta_2 - \frac{8}{9}\epsilon_2 \right] \quad \dots(17)$$

$$y = \pm \frac{\sqrt{3}}{2} \left[1 - \frac{2}{9}\delta_1 - \frac{2}{9}\delta_2 - \frac{4}{9}\epsilon_2 \right] \quad \dots(18)$$

$$\text{Let } x = x_0 + u$$

$$y = y_0 + v$$

Where (x_0, y_0) denotes the coordinates of the equilibrium points under consideration. Let (u, v) denotes the small displacement in x_0, y_0 respectively.

We have the variational equations as

$$\begin{aligned} u'' - 2\alpha v' &= u \Omega_{xx}^o + v \Omega_{xy}^o \\ v'' + 2\alpha u' &= u \Omega_{xy}^o + v \Omega_{yy}^o \end{aligned} \quad \dots\dots(19)$$

We consider only the linear terms in u and v . Here the second partial derivatives of Ω are denoted by subscripts. The subscripts 'o' indicates that the derivatives is to be evaluated at the points (x_0, y_0) under study.

The characteristic equation corresponding to (19) is

$$\begin{aligned} \lambda^4 - (\Omega_{xx}^o + \Omega_{yy}^o - 4\alpha^2)\lambda^2 + \Omega_{xx}^o \Omega_{yy}^o \\ - (\Omega_{xy}^o)^2 = 0 \end{aligned} \quad \dots\dots\dots(20)$$

Now coefficient of λ^2 in the characteristics equation

$$\Omega_{xx}^o + \Omega_{yy}^o - 4\alpha^2 = 3\beta - 4\alpha^2$$

Constant term in the characteristic equation

$$\begin{aligned} \Omega_{xx}^o \Omega_{yy}^o - (\Omega_{xy}^o)^2 \\ = \frac{27}{4} \mu(1-\mu) \left(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2 \right) \end{aligned}$$

Substituting these values in (20), we have

$$\lambda^4 - (3\beta - 4\alpha^2)\lambda^2 + \frac{27}{4}\mu(1-\mu)(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2) = 0 \quad \dots\dots(21)$$

This is a quadratic equation in λ^2 , whose roots are given by

$$\lambda = \pm \left[\frac{(3\beta - 4\alpha^2) \pm \sqrt{D}}{2} \right]^{\frac{1}{2}} \quad \dots\dots(22)$$

Where D is the discriminant of the quadratic equation (21).

$$D = 27(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2)\mu^2 - 27(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2)\mu + (3\beta - 4\alpha^2)^2 \quad \dots\dots\dots(23)$$

$$\text{or, } D = A\mu^2 + B\mu + C \quad \dots\dots\dots(24)$$

Where

$$A = 27(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2)$$

$$B = -27(1 + \frac{2}{9}\delta_1 + \frac{2}{9}\delta_2 + \frac{22}{9}\epsilon_2)\mu$$

$$C = (3\beta - 4\alpha^2)^2 \quad \dots\dots\dots(25)$$

The triangular points are linearly stable if $D > 0$ in the range $0 \leq \mu \leq \frac{1}{2}$.

3. Critical Mass

For the critical mass

$$A\mu^2 + B\mu + C = 0 \quad \dots\dots\dots(26)$$

The critical value of μ in the range of $0 \leq \mu \leq \frac{1}{2}$ is given by

$$\mu_c = \frac{-(B + \sqrt{B^2 - 4AC})}{2A} \quad \dots\dots(27)$$

Now

$$-\frac{B}{2A} = \frac{1}{2}$$

$$\text{and } \frac{\sqrt{B^2 - 4AC}}{2A} = \frac{\sqrt{69}}{18} + \frac{2}{27\sqrt{69}}\delta_1 + \frac{2}{27\sqrt{69}}\delta_2 - \frac{16}{3\sqrt{69}}\epsilon_1 + \frac{76}{27\sqrt{69}}\epsilon_2$$

$$\therefore \mu_c = \frac{1}{2} - \frac{\sqrt{69}}{18} - \frac{2}{27\sqrt{69}}\delta_1 - \frac{2}{27\sqrt{69}}\delta_2 + \frac{16}{3\sqrt{69}}\epsilon_1 - \frac{76}{27\sqrt{69}}\epsilon_2$$

i.e., $0 \leq \mu < \mu_c$.

This shows the range of stability.

We observed that the value of critical mass increases with the increase in mass reduction factor and Coriolis force. But the value of critical mass decreases with the increase in centrifugal force.

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