Common Fixed Point Theorem for Eight Mappings in Menger Space Using Rational Inequality Without Continuity

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Abstract: In this paper we establish Common fixed point theorem for eight mappings in Menger space using the notion of compatibility including rational term without continuity.

Keywords: Menger space, Weak compatibility, Common fixed point.

1. Introduction

Menger space, Weak compatibility, Common fixed point.

2. Preliminaries

Definition 2.1: A probabilistic metric space (PM-space) is an ordered pair (X,F) consisting of a nonempty set X and a function F: X x X -> L, where L is the collection of all distribution functions and the value of F at (u,v) ∈ X x X is represented by F_{u,v}. The function F_{u,v} is assumed to satisfy the following conditions:

(PM-1) F_{u,v}(x) = 1, for all x>0 if and only if u = v

(PM-2) F_{u,v}(0) = 0;

(PM-3) F_{u,v} = F_{v,u};

(PM-4) F_{u,v}(x) = 1 and F_{v,w}(y) = 1 then F_{u,w}(x+y) = 1 for all u,v,w ∈ X and x,y > 0.

Definition 2.2: A mapping t:[0,1] x [0,1] -> [0,1] is called a t-norm if

(a) t(a, 1) = a, t(0, 0) = 0
(b) t(a,b) = t(b,a) (symmetric property)
(c) t(c,d) = t(a,b) for c ≥ a, d ≥ b
(d) t(t(a,b), c) = t(a, t(b, c))

Definition 2.3: A Menger space is a triplet (X,F,t) where (X,F) is a PM-space and t is a t-norm such that the inequality:

F_{u,w}(x+y) ≥ t { F_{u,v}(x), F_{v,w}(y) } for all u,v,w ∈ X and x,y > 0

Definition 2.4: Let {x_n} be a sequence in a Menger space (X,F,t) with continuous t-norm and t(x,x) ≥ x. Suppose for all xe[0,1] there exists kε(0,1) such that for all x>0 and nεN

F_{x_n,x_n+1}(x) ≥ F_{x_{n-1},x_n}(x).

Then {x_n} is a Cauchy sequence in X.

Lemma 1: Let {x_n} be a sequence in a Menger space (X,F,t) with continuous t-norm and t(x,x) ≥ x. Suppose for all xe[0,1] there exists kε(0,1) such that for all x>0 and nεN

F_{x_n,x_{n+1}}(x) ≥ F_{x_{n-1},x_n}(x).

Then {x_n} is a Cauchy sequence in X.

Lemma 2: Let (X,F,t) be a Menger space. If there exists kε(0,1) such that for p,qεX

F_{p,q}(x) ≥ F_{p,q}(x).

Then p=q

In 2006, Bijendra Singh and shishir jain [9] introduced fixed point theorems in Menger space through semi-compatibility and gave the following fixed point theorem for six mappings:

Theorem: Let A,B,S,T,L and M are self mappings on a complete Menger space (X,F,min) satisfying:

(a) L(X) ⊆ ST(X), M(X) ⊆ AB(X)
(b) AB = BA, ST = TS, LB = BL, MT = TM
(c) Either AB or L is continuous.
(d) (L, AB) is semi-compatible and (M, ST) is weak compatible.

(e) There exists kε(0,1) such that
3. Main Results

Theorem (3.1): Let A, B, S, T, L, M and Q be self-mappings on a complete Menger space (X, F, t). Let the functions satisfying:

(3.1.1) \( A(X) \subseteq ST(X) \cap L(X) \cap M(X), B(X) \subseteq PQ(X) \).

(3.1.2) \( PQ = QP, ST = TS, AQ = QA, BT = TB, LT = TL, MT = TM. \)

Let such that \( x_{2n} = TX_{2n+1} = LX_{2n+1} = MX_{2n+1} = y_{2n+1} = BX_{2n+1} = PQX_{2n+2} \) for \( n = 0, 1, 2, \ldots \)

putting \( x = x_{2n} \) and \( y = y_{2n+1} \) for \( x > 0 \) in 3.1.4 then we have

\[
F_{A_{x_{2n}}, B_{x_{2n+1}}} (Kt) \leq \min \{ F_{PQ_{x_{2n}}, L_{x_{2n+1}}} (t), F_{ST_{x_{2n+1}}, L_{x_{2n+1}}} (at), F_{B_{x_{2n+1}}, PQ_{x_{2n+2}}} ((2 - \alpha)t), \]

\[
\left( F_{F_{x_{2n}}, L_{x_{2n+1}}} (t), F_{A_{x_{2n+1}}, PQ_{x_{2n+2}}} ((2 - \alpha)t), \right)\].

Hence

\[
F_{y_{2n}, y_{2n+1}} (Kt) \geq \min \{ F_{y_{2n-1}, y_{2n}} (at), F_{y_{2n}, y_{2n+1}} (at), F_{y_{2n+1}, y_{2n}} (t), F_{y_{2n+1}, y_{2n+2}} (t)\}
\]

Let \( t \in (0, 1) \) and put \( \beta = 1 - \alpha \) we get

\[
F_{y_{2n}, y_{2n+1}} (Kt) \geq \min \{ F_{y_{2n-1}, y_{2n}} (at), F_{y_{2n}, y_{2n+1}} (at), F_{y_{2n+1}, y_{2n}} (t), F_{y_{2n+1}, y_{2n+2}} (t)\}
\]

Making \( \beta \rightarrow 1 \), we get

\[
F_{y_{2n}, y_{2n+1}} (Kt) \geq \min \{ F_{y_{2n-1}, y_{2n}} (t), F_{y_{2n}, y_{2n+1}} (t), F_{y_{2n+1}, y_{2n+2}} (t)\}
\]

Similarly, \( F_{y_{2n+1}, y_{2n+2}} (Kt) \geq \min \{ F_{y_{2n+1}, y_{2n+2}} (t), F_{y_{2n+2}, y_{2n+3}} (t)\} \)

Therefore for all \( n \) even or odd we have

\[
F_{y_{n}, y_{n+1}} (Kt) \geq \min \{ F_{y_{n-1}, y_{n}} (t), F_{y_{n+1}, y_{n+2}} (t)\}
\]

Consequently, it follows that for \( p = 1, 2, 3, \ldots \)

\[
F_{y_{n}, y_{n+1}} (Kt) \geq \min \{ F_{y_{n-1}, y_{n}} (t), F_{y_{n+1}, y_{n+2}} (t)\}
\]

By noting that \( F_{y_{n}, y_{n+1}} (t) \rightarrow 1 \) as \( n \rightarrow \infty \) it follows that

\[
F_{y_{n}, y_{n+1}} (Kt) \geq \min \{ F_{y_{n-1}, y_{n}} (t) \}
\]

Hence by Lemma (1), \( \{ y_{n} \} \) is a Cauchy sequence in \( X \). Now suppose \( PQ(X) \) is complete. Note that the subsequence \( \{ y_{2n+1} \} \) is contained in \( PQ(X) \) call it \( z \). Let \( u \in PQ^{-1} (z) \) then \( PQu = z \). we shall use the fact that subsequence \( \{ y_{2n+1} \} \) also converges to \( z \).

\[
\text{Taking } n \rightarrow \infty \text{we get}
\]

\[
F_{Au, z} (Kt) \geq \min \{ F_{x, z} (t), F_{x, z} (t), F_{z, z} (t), \}
\]

Thus we have

\[
F_{Au, z} (Kt) \geq F_{Au, z} (t)
\]

Therefore by Lemma 2 we have \( Au = z \). since \( PQu = z \) thus we have \( Au = PQu = z \) that is \( u \) is common point of \( A \) and \( PQ \) this proves 3.1.5 (a).

Since \( A(X) \subseteq ST(X) \cap L(X) \cap M(X), Au = z \) implies that \( ST(X) \cap L(X) \cap M(X) \). Then \( STv = Lv = Mv = z \). By putting \( x = x_{2n+2} \) and \( y = v \) with \( \alpha = 1 \) in 3.1.4

\[
\text{Since } A(X) \subseteq ST(X) \cap L(X) \cap M(X), Au = z \text{ implies that } ST(X) \cap L(X) \cap M(X). \text{ Then } STv = Lv = Mv = z. \text{ By putting } x = x_{2n+2} \text{ and } y = v \text{ with } \alpha = 1 \text{ in } 3.1.4
\]
\( F_{Ax_{2n+2,Btn}}(Kt) \geq \min \{ F_{PQ_{2n+2,LP}}(t), F_{ST_{2n+2,LP}}(t), F_{BT_{2n+2,LP}}(t), F_{Az,PR_{2n+2,LP}}(t), F_{Az,PR_{2n+2,LP}}(t), F_{Az,PR_{2n+2,LP}}(t) \} \). 

Taking \( n \to \infty \) we get 
\( F_{Az,Btn}(Kt) \geq \min \{ F_{z,x}(t), F_{x,z}(t), F_{PR_{z,x}}(t), F_{PR_{x,z}}(t), F_{Az,PR_{z,x}}(t), F_{Az,PR_{x,z}}(t) \} \). 

Thus we have 
\( F_{z,Btn}(Kt) \geq F_{z,Btn}(t) \). Therefore by Lemma (2) we have 
\( Bz = z \) since \( STv = Lv = Mt = z \) thus we have \( Bv = STv = Lv = Mt = z \) that is v is coincident point of B and ST, L, M. This proves (b).

The remaining two cases pertain essentially to the previous cases. Indeed if A(X) or B(X) is complete then by 3.1.5 \( z \in A(X) \cap ST(X) \cap M(X) \) or \( z \in B(X) \cap PQ(X) \). Thus 3.1.5 (a) and (b) are completely established. Since the pair \{A, PQ\} is weakly compatible therefore A and PQ commute at their coincidence point that is \( A(PQz) = (PQ/Au = Az = PQz) \).

Since the pair \{B, ST\}, \{L, ST\} and \{L, M\} are weakly compatible therefore 
\( B(STv) = ST(Bv) = z \) and \( Lv = Lv = Mt = z \) that is v is coincident point of B and ST, L, M. Therefore By putting \( x = x_{2n+2} \) and \( y = z \) with \( z = 1 \) in 3.1.4

\( F_{Ax_{2n+2,Btn}}(Kt) \geq \min \{ F_{PQ_{2n+2,LP}}(t), F_{ST_{2n+2,LP}}(t), F_{PR_{PQ_{2n+2,LP}}(t)}, F_{Az,PR_{2n+2,LP}}(t) \} \). 

Taking \( n \to \infty \) we get 
\( F_{Az,Btn}(Kt) \geq \min \{ F_{z,x}(t), F_{x,z}(t), F_{PR_{z,x}}(t), F_{PR_{x,z}}(t), F_{Az,PR_{z,x}}(t), F_{Az,PR_{x,z}}(t) \} \). 

Thus we have 
\( F_{z,Btn}(Kt) \geq F_{z,Btn}(t) \). Therefore by Lemma (2) we have \( Bz = z \) since \( Bz = STz = Lz = Mz = z \). 

By putting \( x = x_{2n+2} \) and \( y = z \) with \( z = 1 \) in 3.1.4

\( F_{Ax_{2n+2,Btn}}(Kt) \geq \min \{ F_{PQ_{2n+2,LP}}(t), F_{ST_{2n+2,LP}}(t), F_{PR_{PQ_{2n+2,LP}}(t)}, F_{Az,PR_{2n+2,LP}}(t) \} \). 

Taking \( n \to \infty \) we get 
\( F_{Az,Btn}(Kt) \geq \min \{ F_{z,x}(t), F_{x,z}(t), F_{PR_{z,x}}(t), F_{PR_{x,z}}(t), F_{Az,PR_{z,x}}(t), F_{Az,PR_{x,z}}(t) \} \). 

Thus we have 
\( F_{z,Btn}(Kt) \geq F_{z,Btn}(t) \). Therefore by Lemma 2 we have \( Tz = z \) since \( STtz = z \) therefore \( Sz = z \). 

By putting \( x = x_{2n+2} \) and \( y = z \) with \( z = 1 \) in 3.1.4

\( F_{Az,Btn}(Kt) \geq \min \{ F_{PQ_{2n+2,LP}}(t), F_{ST_{2n+2,LP}}(t), F_{PR_{PQ_{2n+2,LP}}(t)}, F_{Az,PR_{2n+2,LP}}(t) \} \). 

As \( AQ = QA, PQ = QP \) we have 
\( AQ = QA, PQ = QP \). 

\( F_{z,x}(Kt) \geq \min \{ F_{z,x}(t), F_{z,x}(t), F_{Qz,x}(t), F_{Qz,x}(t), F_{Az,Qz,x}(t), F_{Az,Qz,x}(t) \} \). 

Thus we have 
\( F_{z,Btn}(Kt) \geq F_{z,Btn}(t) \). Therefore by Lemma (2) we have \( Qz = z \). Since \( PQz = z \) therefore \( Pz = z \). By combining the above results we have \( Az = Bz = Lz = Mz = Sz = Tz = Pz = Qz = z \). That is z is a common fixed point of A, B, L, M, S, T, P and Q.

Uniqueness: Let \( z' \neq z' \) be another common fixed point of A, B, L, M, S, T, P and Q, then \( Az' = Bz' = Lz' = Mz' = Sz' = Tz' = Pz' = Qz' = z' \). 

By putting \( x = z \) and \( y = z' \) with \( z = 1 \) in 3.1.4 we have
\[ F_{Ax,Bz}(Kt) \geq \min \{ F_{PQz,Lz}(t), F_{STx,Lx}(t), F_{Az,PQz}(t), \left( \frac{F_{PQz,STx}(t) \cdot F_{Az,PQz}(t)}{F_{PQz,STx}(t) \cdot F_{Az,Lx}(t)} \right) \}. \]

Corollary 3.1.6: Let \( A, S, T, L, M, P \) and \( Q \) are self mappings on a complete Menger space \((X, F, t)\) satisfying:

(1) \( A(X) \subseteq ST(X) \cap L(X) \cap M(X), A(X) \subseteq PQ(X) \).

(2) \( PQ = QP, ST = TS, AQ = QA, AT = TA, LT = TL, MT = TM. \)

(3) \((A, PQ), (L, ST), (A, ST), (L, M)\) are weak compatible.

(4) There exists \( k \in (0, 1) \) such that

\[ F_{Ax,Ay}(Kt) \geq \min \{ F_{PQz,ly}(t), F_{STy,ly}(t), \left( \frac{F_{PQz,STy}(t) \cdot F_{Ax,My}(t)}{F_{PQz,STy}(t) \cdot F_{Ax,ly}(t)} \right) \}. \]

For all \( x, y \in X, \alpha \in (0, 1) \) and \( t > 0 \).

(5). If one of \( A(X), ST(X), PQ(X) \) is a complete sub space of \( X \), then:

(a) \( A \) and \( PQ \) have a coincidence point.

(b) \( A \) and \( ST, L, M \) have a coincidence point.

Then self – maps \( A, S, T, L, M, P \) and \( Q \) have a unique common fixed point in \( X \).

References


