Oscillation Criteria for Third Order Nonlinear Neutral Differential Equations with Deviating Arguments

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***Abstract:** The aim of this paper is to discuss oscillation and asymptotic behavior of a class of third-order nonlinear neutral delay differential equations. A new theorem is presented that improves a number of results reported in the literature. Example is included to illustrate new results.*

Keywords: Oscillation, third order, neutral delay differential equations.

1. Introduction

In this paper we consider third order neutral differential equations of the form

$$\left(r_2(t)\left(\left(r_1(t)(z'(t))^{\alpha_1}\right)'\right)^{\alpha_2}\right)' + \int_a^b q(t,\xi) f(x(g(t,\xi)))d\sigma(\xi) = 0, t \geq t_0, \quad (1.1)$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and the following conditions are satisfied

(A₁) $p, \tau \in C(I, \mathrm{R}), o < p(t) \leq p < 1, \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty, \alpha_1$ and α_2 are a quotient of odd positive integers, $\alpha_1\alpha_2 = \beta$ and $I = [t_0, \infty)$,

(A₂) $r_i \in C(I,(0,\infty)), \int_{t_0}^{\infty} (r_i(t))^{-1/\alpha_i} dt = \infty, i = 1,2,$

(A₃) $f \in C(\mathrm{R},\mathrm{R}), xf(x) > 0$ for $t \geq t_0$,

(A₄) $q \in C(I \times [a,b],[0,\infty)), q(t,\xi)$ is not zero on any half line $[t_\mu,\infty) \times [a,b], t_\mu \geq t_0$,

(A₅) $g \in C(I \times [a,b], \mathrm{R})$, $g(t,\xi) \leq t$ for $t \geq t_0$ and $\xi \in [a,b]$, $g(t,\xi)$ is continuous, has positive partial derivative on $I \times [a,b]$ with respect to t, nondecreasing with respect to ξ and $\lim_{t\to\infty} g(t,\xi) = \infty$,

(A₆) $\sigma \in C([a,b],\mathrm{R})$, σ is nondecreasing and the integral of Eq. (C-1) is in the sense Riemann-stieltijes.

We mean by a solution of Eq. (1.1) a function $x(t):[t_x,\infty) \to \mathrm{R}, t_x \geq t_0$ such that $x(t), r_1(t)(z'(t))^{\alpha_1}$ and $r_2(t)((r_1(t)(z'(t))^{\alpha_1})')^{\alpha_2}$ are continuously differentiable for all $t \in [t_x,\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq t_x$. A solution of Eq. (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

Asymptotic properties of solutions of differential equations of the second and third order have been subject of intensive studying in the literature. This problem for neutral differential equations has received considerable attention in recent years (see [1] - [12]).

The aim of this paper is to discuss asymptotic behavior of solutions of class of third order neutral delay differential equation. By using Riccati transformation technique , we established sufficient conditions which insure that solution of class of third order neutral delay differential equation is oscillatory or tended to zero. The results of this study basically generalize and improve the previous results.

Following [Philos [13]], we introduce a class of functions $\Im$ as follows. Let

$$D_0 = \{(t,s) : t > s > t_0\} \text{ and } D = \{(t,s) : t \geq s \geq t_0\}$$

A kernel function $H \in C(D,\mathbb{R})$ is said to belong to the function class $\Im$, written by $H \in \Im$, if

1) $H(t,s) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on D_0

2) $H(t,s)$ has a continuous and nonpositive partial derivative $\partial H/\partial s$ on D_0 such that the condition

$$\frac{\partial H(t,s)}{\partial s} = -h(t,s)H(t,s) \text{ for all } (t,s) \in D_0$$

is satisfied for some $h \in C(D,\mathbb{R})$.

Let us state two sets of conditions commonly used, which we rely on:

(S_1) $\frac{f'(x)}{|f(x)|^{\frac{1-\beta}{\beta}}} \geq k_1 > 0$ for $x \neq 0$ and

$-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$.

(S_2) $\frac{f(x)}{x^\beta} \geq k > 0$ for $x \neq 0$.

For the sake of convenience, we introduce the following notation:

$$E_0(z(t)) = z(t),\ E_i(z(t)) = r_i(t)\left(\frac{d}{dt}E_{i-1}(z(t))\right)^{\alpha_i}, i = 1,2,$$

$$R(t,t_0) = \left(\frac{1}{r_1(t)}\int_{t_0}^{t}\frac{1}{r_2^{1/\alpha_2}(s)}ds\right)^{1/\alpha_1}, \quad \mu = \frac{\beta^\beta}{(\beta+1)^{\beta+1}}$$

and

$$\hat{q}(t) = \int_a^b q(t,\xi)d\sigma(\xi)$$

and let there exists a function $\rho \in C(I,\mathbb{R}^+)$ such that

$$Q(t,s) = \left|\frac{\rho'(s)}{\rho(s)} - h(t,s)\right|.$$

2. Several Lemmas

We begin with some useful lemmas, which we intend to use later.

Lemma 2.1. *Assume that* $f(y) = Uy - Vy^{\frac{\eta+1}{\eta}}$, *where* U *and* V *are constants,* $V > 0$ *and* η *is a quotient of odd positive integers. Then* f *attends its maximum value on* $\mathbb{R}$ *at* $y^* = \left(\frac{U\eta}{V(\eta+1)}\right)^\eta$ *and*

$$\max_{y\in\mathbb{R}} f = f(y^*) = \frac{\eta^\eta}{(\eta+1)^{\eta+1}}U^{\eta+1}V^{-\eta}.$$

Lemma 2.2. *Let* $x(t)$ *be a positive solution of Eq. (1.1). Then* $z(t)$ *has only one of the following two properties eventually:*

$$(P_1)\ z(t) > 0, z'(t) > 0 \text{ and } \frac{d}{dt}E_1(z(t)) > 0,$$

$$(P_2)\ z(t) > 0, z'(t) < 0 \text{ and } \frac{d}{dt}E_1(z(t)) > 0.$$

Proof. Let $x(t)$ be a positive solution of Eq. (1.1). From (A_1) and (A_5), there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$ and $x(g(t,\xi)) > 0$ for $t \geq t_1$. Then $z(t) > 0$ and Eq. (C-1) implies that $\frac{d}{dt}E_2(z(t)) \leq 0$. Hence, $E_2(z(t))$ is a non-increasing function and of one sign. We claim that $E_2(z(t)) > 0$ for $t \geq t_1$. Suppose that $E_2(z(t)) < 0$ for $t \geq t_2 \geq t_1$, then there exists a $t_3 \geq t_2$ and constant $K_1 > 0$ such that

$$\frac{d}{dt}E_1(z(t)) < -K_1(r_2(t))^{-1/\alpha_2},$$

for $t \geq t_3$. By integrating the last inequality from t_3 to t, we get

$$E_1(z(t)) < E_1(z(t_3)) - K_1\int_{t_3}^{t}(r_2(s))^{-1/\alpha_2}ds.$$

Letting $t \to \infty$, from (A_2), we have $\lim_{t\to\infty} E_1(z(t)) = -\infty$. Then there exists a $t_4 \geq t_3$ and constant $K_2 > 0$ such that

$$z'(t) < -K_2(r_1(t))^{-1/\alpha_1},$$

for $t \geq t_4$. By integrating this inequality from t_4 to t and using (A_2), we get $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Now we have $E_2(z(t)) > 0$ for $t \geq t_1$. Therefore, $E_1(z(t))$ is increasing function. Thus (P_1) or (P_2) holds for $z(t)$, eventually.

Lemma 2.3. *Let* (S_1) *holds,* $x(t)$ *be a positive solution of Eq. (1.1), and* $z(t)$ *has the property* (P_2). *Assume that*

$$\int_{t_0}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}\widehat{q}(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1}dv=\infty. \tag{2.1}$$

Then the solution $x(t)$ *is converges to zero as* $t\to\infty$.

Proof Let $x(t)$ be a positive solution of Eq. (C-1). Since $z(t)$ satisfies the property (P_2), we get

$$\lim_{t\to\infty} z(t)=\gamma.$$

Now. We shall prove that $\gamma=0$. Let $\gamma>0$, then we have $\gamma<z(t)<\gamma+\varepsilon$ for all $\varepsilon>0$ and t enough large. Choosing $\varepsilon<\frac{1-p}{p}\gamma$, we obtain

$$\begin{aligned} x(t) &= z(t)-p(t)x(\tau(t)) \\ &> \gamma - pz(\tau(t)) \\ &> L(\gamma+\varepsilon) > Lz(t), \end{aligned}$$

where $L=\frac{\gamma-p(\gamma+\varepsilon)}{\gamma+\varepsilon}>0$. Hence, from (1.1), (S_1) and (A_5), we have

$$\begin{aligned} \frac{d}{dt}E_2(z(t)) &< -kL^{\beta}\int_a^b q(t,\xi)z^{\beta}(g(t,\xi))d\sigma(\xi) \\ &< -kL^{\beta}z^{\beta}(t)\widehat{q}(t) \\ &< -kL^{\beta}\gamma^{\beta}\widehat{q}(t). \end{aligned}$$

By integrating two times from t to ∞, we get

$$-z'(t)>C\left(\frac{1}{r_1(t)}\int_t^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}\widehat{q}(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1},$$

where $C=k^{1/\beta}L\gamma>0$. Integrating the last inequality from t_1 to ∞, we have

$$z(t_1)>C\int_{t_1}^{\infty}\left(\frac{1}{r_1(v)}\int_v^{\infty}\left(\frac{1}{r_2(u)}\int_u^{\infty}\widehat{q}(s)ds\right)^{1/\alpha_2}du\right)^{1/\alpha_1}dv.$$

This contradicts to the condition (2.1), then $\lim_{t\to\infty} z(t)=0$, which implies that $\lim_{t\to\infty} x(t)=0$.

Lemma 2.4. *Let* (S_2) *holds,* $x(t)$ *be a positive solution of Eq. (C-1) and* $z(t)$ *has the property* (P_2). *If the condition (C-3) holds, then the solution* $x(t)$ *is converges to zero as* $t\to\infty$.

Proof Proceeding as in the proof of Lemma 2.3. Hence, from (1.1), (S_2) and (A_5), we have

$$\begin{aligned} \frac{d}{dt}E_2(z(t)) &\le -\int_a^b q(t,\xi)f(Lz(g(t,\xi)))d\sigma(\xi) \\ &< -f(L)f(\gamma)\widehat{q}(t). \end{aligned}$$

The rest of the proof runs as in Lemma 2.4.

3. Oscillation Theorems

Theorem 3.1. *Let* (S_1) *and (2.1) hold. If there exist functions* $\rho\in C(I,\mathbb{R}^+)$ *and* $H\in\Im$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)(\Theta_1(s)-\mu Q^{\beta+1}(t,s)l_1(s))ds=\infty, \tag{3.1}$$

where

$$\Theta_1(t)=\rho(t)\int_a^b q(t,\xi)f(1-p(g(t,\xi)))d\sigma(\xi),$$

and

$$l_1(t)=\rho(t)(k_1R(g(t,a),t_0)g'(t,a))^{-\beta}.$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Proof Let x be a non-oscillatory solution of Eq. (1.1) on I. Without loss of generality we assume that $x(t)\neq 0$ for $t\ge t_0$. Futhermore, we suppose that $x(t)>0$ for $t\ge t_0$. Note that (A_1) and (A_5), there exists a $T_0\ge t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(g(t,\xi))>0$ for $t\ge T_0$. By Lemma C-L2, we have that $z(t)$ has the property (P_1) or the property (P_2).

If $z(t)$ has the property (P_2). Since (2.1) hold, the conditions in Lemma 2.3 are satisfied. Hence, we obtain

$$\lim_{t\to\infty} x(t) = 0$$

Now, Let $z(t)$ satisfies the property (P_1), then we have

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1 - p(t))z(t) \tag{3.2}$$

Thus, from (1.1), (S_1) and (A_5), we have

$$\frac{d}{dt}E_2(z(t)) \leq -f\big(z(g(t,a))\big)\int_a^b q(t,\xi)\, f\big(1-p(g(t,\xi))\big)d\sigma(\xi).$$

We define

$$\omega(t) = \rho(t)\frac{E_2(z(t))}{f\big(z(g(t,a))\big)}.$$

By differentiating, we get

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \Theta_1(t)$$
$$-\rho(t)\frac{E_2(z(t))}{f^2\big(z(g(t,a))\big)} f'\big(z(g(t,a))\big)z'(g(t,a))g'(t,a).$$

From (P_1), we have

$$E_1(z(t)) = E_1(z(t_1)) + \int_{t_1}^t \frac{E_2^{1/\alpha_2}(z(s))}{r_2^{1/\alpha_2}(s)}ds$$
$$\geq E_2^{1/\alpha_2}(z(t)) \int_{t_1}^t \frac{1}{r_2^{1/\alpha_2}(s)}ds,$$

for $t \geq t_1 \geq T_0$. Since $\frac{d}{dt}E_2(z(t)) \leq 0$, we obtain

$$z'(g(t,a)) \geq E_2^{1/\beta}(z(t))R(g(t,a),t_1).$$

Hence,

$$\omega'(s) \leq -\Theta_1(s) + \frac{\rho'(s)}{\rho(s)}\omega(s) - l_1^{-1/\beta}(s)\omega^{\frac{\beta+1}{\beta}}(s), \tag{3.3}$$

for $s \geq t_1$. Multiplying relation (3.3) by $H(t,s)$ and integrating from t_1 to t, we get

$$\int_{t_1}^t H(t,s)\Theta_1(s)ds \leq -\int_{t_1}^t H(t,s)\omega'(s)ds + \int_{t_1}^t H(t,s)\frac{\rho'(s)}{\rho(s)}\omega(s)ds$$
$$-\int_{t_1}^t H(t,s)l_1^{-1/\beta}(s)\omega^{\frac{\beta+1}{\beta}}(s)ds$$
$$= H(t,t_1)\omega(t_1) + \int_{t_1}^t H(t,s)\Big(Q(t,s)\omega(s) - l_1^{-1/\beta}\omega^{\frac{\beta+1}{\beta}}\Big)ds$$

If $\eta = \beta, U = Q, V = l_1^{-1/\beta}$ and $X = \omega$, then from Lemma 2.1, we obtain

$$Q\omega - l_1^{-1/\beta}\omega^{\frac{\beta+1}{\beta}} \leq \mu Q^{\beta+1}l_1.$$

Therefore, we get

$$\omega(t_2) \geq \frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)(\Theta_1(s) - \mu Q^{\beta+1}(t,s)l_1(s))ds.$$

which is contrary to (3.1). This completes the proof of Theorem 3.1.

Theorem 3.2. *Let* (S_2) *and (2.1) hold. If there exist functions* $\rho \in C(I,\mathbb{R}^+)$ *and* $H \in \Im$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)(\Theta_2(s) - \mu Q^{\beta+1}(t,s)l_2(s))ds = \infty, \tag{3.4}$$

where

$$\Theta_2(t) = k_2\rho(t)(1-p)^\beta \hat{q}(t),$$

and

$$l_1(t) = \rho(t)(\beta R(g(t,a),t_0)g'(t,a))^{-\beta}.$$

Then every solution of Eq. (1.1) is either oscillatory or tends to zero as $t\to\infty$.

Proof Let $x(t)$ is an eventually positive solution of equation (1.1) on I. As in the proof of Theorem C-T1, we have that $z(t)$ has the property (P_1) or the property (P_2). Let $z(t)$ has the property (P_2). From Lemma 3.4, we obtain $\lim_{t\to\infty} x(t) = 0$. On the other hand, when

$z(t)$ satisfies the property (P_1), we have that (3.2) holds. Thus, from (1.1), (S_1) and (A_5), we get

$$\frac{d}{dt}E_2(z(t)) \leq -k_2(1-p)^\beta z^\beta(g(t,a))\int_a^b q(t,\xi)d\sigma(\xi).$$

We define

$$\omega(t) = \rho(t)\frac{E_2(z(t))}{z^\beta(g(t,a))}.$$

By differentiating, we get

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) - \Theta_2(t) - \beta\rho(t)\frac{E_2(z(t))}{z^{\beta+1}(g(t,a))}z'(g(t,a))g'(t,a).$$

Next, by following the same steps in the proof of Theorem 3.1. Hence,

$$\omega'(s) \leq -\Theta_2(s) + \frac{\rho'(s)}{\rho(s)}\omega(s) - l_2^{-1/\beta}(s)\omega^{\frac{\beta+1}{\beta}}(s),$$

The rest of the proof runs as in Theorem 3.1. The proof is complete.

Remark *If* $\alpha_1 = \alpha_2 = 1$, $\tau(t) = t-\tau$ *and* $f(x) = x$. *Then, Theorem 3.1 extend Theorem 2.1 in Candan [3].*

Remark *If* $\alpha_1 = \alpha_2 = 1$, $a = 0, b = 1$, $q(t,\xi) = q(t), g(t,\xi) = g(t)$ *and* $f(x) = x$. *Then, Theorem 3.2. extend and improve Theorem 2.1 in Li [9].*

Theorem 3.3. *Let* (S_1) *and (2.1) hold. Assume that there exist functions* $\rho \in C(I,\mathbb{R}^+)$ *and* $H \in \Im$ *such that for every* $T \geq t_0$,

$$0 < \inf_{s\geq T}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T)}\right] \leq \infty \tag{3.5}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \mu H(t,s)Q^{\beta+1}(t,s)l_1(s)ds < \infty, \tag{3.6}$$

hold. If there exists a function $\psi \in C(I,\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left(\frac{\psi_+^{\beta+1}(s)}{l_1(s)}\right)^{1/\beta}ds = \infty, \tag{3.7}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)(\Theta_1(s) - \mu Q^{\beta+1}(t,s)l_1(s))ds \geq \psi(T), \tag{3.8}$$

where $\psi_+(s) = \max\{\psi(s),0\}$. *Then every solution of Eq. (1.1) is either oscillatory or tends to zero as* $t\to\infty$.

Proof As the proof of Theorem 3.1, we can see that (3.3) holds. It follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)(\Theta_1(s) - \mu Q^{\beta+1}(t,s)l_1(s))ds$$

$$\leq \omega(t_1) - \liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)(\Phi(t,s) + \mu Q^{\beta+1}(t,s)l_1(s))ds,$$

where $\Phi(t,s) = l_1^{-1/\beta}(s)\omega^{\frac{\beta+1}{\beta}}(s) - Q(t,s)\omega(s)$. From (3.8), we have

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)(\Phi(t,s) + \mu Q^{\beta+1}(t,s)l_1(s))ds$$

$$\leq \omega(t_1) - \psi(t_1) < \infty.$$

Now, we define functions

$$F_1(t) = \frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)Q(t,s)\omega(s)ds$$

and

$$F_2(t) = \frac{1}{H(t,t_1)}\int_{t_1}^t H(t,s)l_1^{-1/\beta}(s)\omega^{\frac{\beta+1}{\beta}}(s)ds$$

From (3.6), we obtain

$$\liminf_{t\to\infty} (F_2(t) - F_1(t)) < \infty$$

The remainder of the proof is similar to the theorem 3.1 given in [6] and hence is omitted.

Theorem 3.4. *Let* (S_2) *and (2.1) hold. Assume that there exist functions* $\rho \in C(I, \mathbb{R}^+)$ *and* $H \in \Im$ *such that for every* $T \geq t_0$, *(C-15) and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \mu H(t,s) Q^{\beta+1}(t,s) l_2(s) ds < \infty,$$

hold. If there exists a function $\psi \in C(I, \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left(\frac{\psi_+^{\beta+1}(s)}{l_2(s)} \right)^{1/\beta} ds = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)(\Theta_2(s) - \mu Q^{\beta+1}(t,s) l_2(s)) ds \geq \psi(T),$$

where $\psi_+(s) = \max\{\psi(s), 0\}$ *. Then every solution of Eq. (1.1) is either oscillatory or tends to zero as* $t \to \infty$.

Example *Consider the third order neutral delay differential equation*

$$\left(\frac{1}{t} \left(\left(t (z'(t))^3 \right)' \right)^{1/3} \right)' + \int_{1/2}^1 \frac{\gamma e^{t(1-\xi)}}{e^{t/2} - 1} x(\xi t) d\xi = 0,$$

where $z(t) = x(t) + \frac{1}{2} x(\frac{t}{2})$ *and* $t > 0.$ *Choose* $\rho(t) = 1, H(t,s) = (t-s)^2$ *and* $k = 1$ *. Hence, we get*

$$\Theta_1(t) = \frac{\gamma}{2t} \text{ and } l_1(t) = \frac{4^{4/3}}{t}$$

It is easy to see that the conditions (2.1) and (3.1) are hold. Then, from Theorem 3.1, every nonoscillatory solution of this equation tends to zero as $t \to \infty$.

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