Common Fixed Point Results in b-metric-like Spaces

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Abstract: Fixed point theory is a powerful tool in mathematics. In this paper, we introduce a common fixed point results in new generalize b-metric-like spaces. This result extend and generalize many existing results in the literature.

Keywords: fixed point

1. Introduction

The concept of b-metric space was introduced and studied by Czerwik [3]. Since then several papers have been dealt with fixed point theory for single-valued and multivalued operators in b-metric spaces. Amini-Harandi [2] introduced the notion of metric-like space, which is an interesting generalization of partial metric space. Recently, Mohammed Ali Algamdi [1] introduced a new generalization of metric-like space and partial metric space is called a b-metric-like space and studied some fixed point theorems of b-metric-like space. These results improved some well-known results in the literature.

2. Preliminaries

In this section, we recall some of the metric spaces and mappings as follows:

Definition 2.1. [1] A b-metric-like on a non empty set X is a function \( \vartheta : X \times X \to [0, +\infty) \) such that for all \( p, q, r \in X \) and a constant \( K \geq 1 \) the following three conditions hold true:

\[
\begin{align*}
(1) & \quad \vartheta(p, q) = 0 \Rightarrow p = q \\
(2) & \quad \vartheta(p, q) = \vartheta(q, p) \\
(3) & \quad \vartheta(p, q) \leq K(\vartheta(p, r) + \vartheta(r, q))
\end{align*}
\]

The pair \((X, \vartheta)\) is called a b-metric-like space.

Definition 2.2. [1] Let \( X = [0, +\infty) \). Define the function \( \vartheta : X^2 \to X \) by \( \vartheta(p, q) = (p + q)^2 \). Then \((X, \vartheta)\) is a b-metric-like space with constant \( K = 2 \). Clearly, \((X, \vartheta)\) is not a b-metric or metric-like space. Indeed, for all \( p, q, r \in X \)

\[
\begin{align*}
\vartheta(p, q) = (p + q)^2 & \leq (p + r + r + q)^2 \\
& = (p + r)^2 + (r + q)^2 + 2(p + r)(r + q) \\
& \leq 2[(p + r)^2 + (r + q)^2] \\
& = 2[\vartheta(p, r) + \vartheta(r, q)]
\end{align*}
\]

and so (D3) holds. Clearly, (D1) and (D2) hold.

Definition 2.3. An element \((a, b) \in X \times X\) is called a coupled fixed point of \( T : X \times X \to X \) if \( a = T(a, b) \) and \( b = T(b, a) \).

Definition 2.4. An element \((a, b) \in X \times X\) is called a coupled coincidence point \( S, T : X \times X \to X \) if \( S(a, b) = T(a, b) \) and \( S(b, a) = T(b, a) \).

Example 2.5. Let \( X = \mathbb{N} \) and \( S, T : X \times X \to X \) defined as \( S(a, b) = a + b - ab + \sin(a + b) \) and \( T(a, b) = a + b + \cos(a + b) \) for all \( a, b \in X \). Then \((\frac{\pi}{4}, \pi)\) and \((\frac{\pi}{4}, 0)\) are coupled coincidence points of \( S \) and \( T \).

Definition 2.6. An element \((a, b) \in X \times X\) is called a \( S, T \) if \( a = S(a, b) = T(a, b) \) and \( b = S(b, a) = T(b, a) \).

Example 2.7. Let \( X = \mathbb{N} \) and \( S, T : X \times X \to X \) defined as \( S(a, b) = ab \) and \( T(a, b) = a + (b - a)^2 \) for all \( a, b \in X \). Then \((0, 0)\) and \((1, 1)\) are common coupled fixed points of \( S \) and \( T \).

3. Main Results

Theorem 3.1. Let \((X, \vartheta)\) be a complete b-metric-like space and a constant \( K \geq 1 \) and let the mapping \( S, T : X \times X \to X \) satisfy

\[
\begin{align*}
\vartheta(S(a, b), T(u, v)) & \leq a \frac{\vartheta(S(a, u), T(u, u))}{2} + b \frac{\vartheta(a, S(a, b))}{(1 + \vartheta(u, T(u, v)))} + \gamma \frac{\vartheta(u, T(u, v))}{(1 + \vartheta(a, T(u, v)))}
\end{align*}
\]

for all \( a, b, u, v \in X \) and \( a, b, \gamma \geq 0 \) with \( K \alpha + \beta < 1 \) and \( a + \gamma < 1 \). Then \( S \) and \( T \) have a unique common coupled fixed point in \( X \).

Proof. Step 1: Firstly, We show that \( a_n, b_n \) are Cauchy sequences in \( X \).

Let \( a_0, b_0 \in X \) be any arbitrary points. Define \( a_{2k+1} = S(a_{2k}, b_{2k}), b_{2k+1} = S(b_{2k}, a_{2k}) \) and \( a_{2k+1} = T(a_{2k+1}, b_{2k+1}), b_{2k+2} = T(b_{2k+1}, a_{2k+1}) \) for \( k = 0, 1, 2, 3, \ldots \).
Now, if
\[ \vartheta(a_{n+1}, a_n) + \vartheta(b_n, b_{n+1}) = \delta_n, \text{then } \delta_n, \text{then } \delta_n \leq h \delta_{n-1} \leq - - - \leq h^n \delta_0 \]

For \( m > n \), we have
\[ (\vartheta(a_m, a_n) + \vartheta(b_m, b_n)) \leq K(\vartheta(a_n, a_{n+1}) + \vartheta(b_n, b_{n+1})) \]
\[ \leq Kh^n \delta_0 + K^2 h^n+1 \delta_0 + ... + K^{m-n} h^{m-1} \delta_0 \]
\[ \leq Kh^n[1 + (K + (K))^2 + ... + \delta_0) \]
\[ = \frac{1}{1 - Kh} \delta_0 \to 0 \text{ as } n \to \infty \]

This shows that \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences in \( X \). Since \( X \) is a complete b-metric-like space, there exists \( a, b \in X \) such that \( a_n \to a \) and \( b_n \to b \).

Step 2: Now, we show that \( a = S(a, b) \) and \( b = S(b, a) \).

We suppose on the contrary that \( a \neq S(a, b) \) and \( b \neq S(b, a) \), so that
\[ \vartheta(a, S(a, b)) = l_1 > 0 \text{ and } \vartheta(b, S(b, a)) = l_2 > 0 \]

Consider
\[ l_1 = \vartheta(a, S(a, b)) \leq K[\vartheta(a, a_{n+2}) + \vartheta(a_{n+2}, S(a, b))] \]
\[ \leq K\vartheta(a, a_{n+2}) + K\vartheta(T(a_{n+1}, b_{n+2}), S(a, b)) \]
\[ = K\vartheta(a, a_{n+2}) + Ka \frac{\vartheta(a_{n+2}, b_{n+1})}{2} \]
\[ + K \beta \left[ \vartheta(a, S(b, a)) \right] \frac{\vartheta(a_{n+2}, b_{n+1})}{2} + K \gamma \frac{\vartheta(a_{n+2}, S(b, a))}{2} \]
\[ \leq K\vartheta(a, a_{n+2}) + Ka \frac{\vartheta(a_{n+2}, b_{n+1})}{2} \]

By taking \( k \to \infty \), we get
\[ l_1 = 0, \text{ which is contradiction.} \]

Therefore, \( \vartheta(a, S(a, b)) = 0 \). This implies \( a = S(a, b) \).

Similarly, we can prove that \( b = S(b, a) \).

It follows similarly we can show that \( a = T(a, b) \) and \( T(b, a) \).

So we have proved that \( (a, b) \) is a common coupled fixed point of \( S \) and \( T \).

Step 3: We now show that \( S \) and \( T \) have a unique common coupled fixed point.

Let \((a^*, b^*) \in X \times X \) be another common coupled fixed point of \( S \) and \( T \). Then
\[ \vartheta(a^*, a^*) = \vartheta(S(a, b), T(a', b')) \]
\[ \leq \vartheta(a, a^*) + \vartheta(b, b^*) \]
\[ + \beta \frac{\vartheta(a, S(a, b)) + \vartheta(b, S(b, a))}{2} \]
\[ + \gamma \frac{\vartheta(a, T(a, b)) + \vartheta(b, T(b, a))}{2} \]

Continuing in this way,
\[ \vartheta(a_{n+1}, a_n) + \vartheta(b_{n+1}, b_n) \leq h^n \vartheta(a_0, a_1) + \vartheta(b_0, b_1) \]

Similarly, we can easily prove that
\[ \vartheta(b, b^*) \leq \frac{\alpha}{(2 - \alpha - 2\gamma)} \vartheta(a, a^*) \]

Adding, we get

\[ \vartheta(a, a^*) + \vartheta(b, b^*) \leq \frac{\alpha}{(2 - \alpha - 2\gamma)} \left[ \vartheta(a, a^*) + \vartheta(b, b^*) \right]. \]

\[ (2 - 2\alpha - 2\gamma) \left[ \vartheta(a, a^*) + \vartheta(b, b^*) \right] \leq 0 \]

\[ \vartheta(a, a^*) + \vartheta(b, b^*) = 0. \]

This implies, \( a = a^* \) and \( b = b^* \)

**References**


