Common Fixed Point Results in b-metric-like Spaces

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Abstract: ***Fixed point theory is a powerful tool in mathematics. In this paper, we introduce a common fixed point results in new generalize b-metric-like spaces. This result extend and generalize many existing results in the literature.***

Keywords: fixed point

1. Introduction

The concept of b-metric space was introduced and studied by czerwik [3]. since then serval papers have been dealt with fixed point theory for single-valued and multivalued operators in b-metric spaces. Amini-Harandi [2] introduced the notion of metric-like space, which is an interesting generalization of partial metric space. Recently, Mohammed Ali Algamdi [1] introduced a new generalization of metric-like space and partial metric space is called a b-metric-like space and studied some fixed point theorem of b-metric-like space. These results improved some well-known results in the literature.

2. Preliminaries

In this section, we recall some of the metric spaces and mappings as follows:

Definition 2.1. [1] A b-metric-like on a non empty set X is a function $\vartheta : X \times X \rightarrow [0, +\infty)$ such that for all $p, q, r \in X$ and a constant K $\geq$ 1 the following three conditions hold true :

$(D1)$ $if\ \vartheta(p,q) = 0 \Rightarrow p = q$
$(D2)$ $\vartheta(p,q) = \vartheta(q,p)$
$(D3)$ $\vartheta(p,q) \leq K(\vartheta(p,r) + \vartheta(r,q))$
The pair (X,ϑ) is called a b-metric-like space.

Example 2.2. [1] *Let* $X = [0, +\infty)$. *Define the function* $\vartheta : X^2 \rightarrow [0,+\infty)$ *by* $\vartheta(p,q) = (p+q)^2$. *Then* (X,ϑ) *is a b – metric – like space with constant* $K = 2$. *Clearly* (X,ϑ) *is not a b – metric or metric – like space. Indeed, for all* $p, q, r \in X$

$$\vartheta(p,q) = (p+q)^2 \leq (p+r+r+q)^2 = (p+r)^2 + (r+q)^2 + 2(p+r)(r+q) \leq 2[(p+r)^2 + (r+q)^2] = 2(\vartheta(p,r) + \vartheta(r,q))$$

and so $(D3)$ *holds. Clearly,* $(D1)$ *and* $(D2)$ *hold.*

Definition 2.3. An element $(a,b) \in X \times X$ is called a coupled fixed point of $T : X \times X \rightarrow X$ if $a = T(a,b)$ *and* $S(b,a) = T(b,a)$.

Definition 2.4. An element $(a,b) \in X \times X$ is called a coupled Coincidence point $S,T : X \times X \rightarrow X$ if $S(a,b) = T(a,b)$ and $S(b,a) = T(b,a)$.

Example 2.5.
Let $X = \Re$ *and* $S,T : X \times X \rightarrow X$ *defined as* $S(a,b) = a + b - ab + \sin(a+b)$ *and* $T(a,b) = a + b + \cos(a+b)$ *for all* $a,b \in X$. *Then* $\left(0,\frac{\pi}{4}\right)$ *and* $\left(\frac{\pi}{4},0\right)$ *are coupled coincidence points of* S *and* T.

Definition 2.6. An element $(a,b) \in X \times X$ *is called a of* $S,T : X \times X \rightarrow X$ *if* $a = S(a,b) = T(a,b)$ *and* $b = S(b,a) = T(b,a)$.

Example 2.7.
Let $X = \Re$ *and* $S,T : X \times X \rightarrow X$ *defined as* $S(a,b) = ab$ *and* $T(a,b) = a + (b-a)^2$ *for all* $a,b \in X$. *Then* $(0,0)$ *and* $(1,1)$ *are common coupled fixed points of* S *and* T.

3. Main Results

Theorem 3.1. *Let* (X,ϑ) *be a complete b – metric – like slpace and a constant* $K \geq 1$ *and let the mapping* $S,T : X \times X \rightarrow X$ *satisfy*

$$\vartheta(S(a,b),T(u,v)) \leq \alpha \frac{\vartheta(a,u) + \vartheta(b,u)}{2} + \beta \frac{\vartheta(a,S(a,b))\,\vartheta(u,v))}{(1+\vartheta(a,u) + \vartheta(b,u))} + \gamma \frac{\vartheta(u,S(a,b))\vartheta(a,T(u,v))}{(1+\vartheta(a,u)+\vartheta(b,u))} \quad (3.1)$$

for all $a,b,u,v \in X$ *and* $\alpha, \beta \geq 0$ *with* $K\alpha + \beta < 1$ *and* $\alpha + \gamma < 1$. *Then S and T have a unique common coupled fixed point in X.*

Proof. **Step 1** : Firstly, We show that a_n, b_n are Cauchy sequence in X.

Let $a_0, b_0 \in X$ be any arbitrary points. Define $a_{2k+1} = S(a_{2k}, b_{2k}), b_{2k+1} = S(b_{2k}, a_{2k})$ and $a_{2k+1} = T(a_{2k+1}, b_{2k+1}), b_{2k+2} = T(b_{2k+1}, a_{2k+1})$ for $k = 0,1,2,3, \ldots \ldots$

Now

$\vartheta(a_{2k+1},a_{2k+2}) = \vartheta(S(a_{2k},b_{2k}),T(a_{2k+1},b_{2k+1}))$

$\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})+\vartheta(b_{2k},b_{2k},b_{2k+1})}{2} + \beta\frac{\vartheta(a_{2k},S(a_{2k},b_{2k}))\vartheta(a_{2k+1},T(a_{2k+1},b_{2k+1}))}{(1+\vartheta(a_{2k},a_{2k+1})+\vartheta(b_{2k},b_{2k+1}))} +$

$\gamma\frac{\vartheta\,(a_{2k+1},S(a_{2k},b_{2k}))\vartheta\,(a_{2k},T\,(a_{2k+1},b_{2k+1}))}{(1+\vartheta\,(a_{2k},a_{2k+1})+\,\vartheta(b_{2k},b_{2k+1}))}\,\vartheta$

(a_{2k+1},a_{2k+2})

$= \alpha\frac{\vartheta(a_{2k},a_{2k+1})+\,\vartheta(b_{2k},b_{2k+1})}{2} + \beta\frac{\vartheta(a_{2k},a_{2k+1})\vartheta(a_{2k+1},a_{2k+1},a_{2k+2})}{(1+\vartheta(a_{2k},a_{2k+1})+\,\vartheta\,(b_{2k},b_{2k+1}))} +$

$\gamma\frac{\vartheta(a_{2k+1},a_{2k+1})\,\vartheta\,(a_{2k},a_{2k+2})}{(1+\vartheta(a_{2k},a_{2k+1})+\vartheta(b_{2k},b_{2k+1}))}$

$\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})+\vartheta(b_{2k},b_{2k+1})}{2} + \beta\frac{\vartheta(a_{2k},a_{2k+1})\vartheta(a_{2k+1}\,,a_{2k+2})}{(1+\vartheta(a_{2k},a_{2k+1})+\vartheta(b_{2k},b_{2k+1}))} +$

$\gamma(2\vartheta(a_{2k+1},a_{2k+2}))$

$\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})}{2} + \alpha\frac{\vartheta(b_{2k},b_{2k+1}}{2} + \beta\vartheta(a_{2k+1},a_{2k+2}) + \gamma(2\vartheta(a_{2k+1},a_{2k+2}))$

$(1-\beta-2\gamma)\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})}{2} + \alpha\frac{\vartheta(b_{2k},b_{2k+1})}{2}$

$\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})}{2(1-\beta-2\gamma)} + \alpha\frac{\vartheta(b_{2k},b_{2k+1})}{2(1-\beta-2\gamma)}$

$\vartheta(a_{2k+1},a_{2k+2}) \le \alpha\frac{\vartheta(a_{2k},a_{2k+1})}{2(1-\beta)} + \alpha\frac{\vartheta(b_{2k},b_{2k+1})}{2(1-\beta)}$

Similarly

$\vartheta(b_{2k+1},b_{2k+2}) \le \alpha\frac{\vartheta(b_{2k},b_{2k+1})}{2(1-\beta)} + \alpha\frac{\vartheta(a_{2k},a_{2k+1})}{2(1-\beta)}$

Common Coupled Fixed Point Theorems

Add (3.2) and (3.3)

$[\vartheta(a_{2k+1},a_{2k+2}) + \vartheta(b_{2k+1},b_{2k+2})] \le \frac{\alpha}{(1-\beta)}[\vartheta(a_{2k},a_{2k+1}) + \vartheta(b_{2k},b_{2k+1})]$

$= h[\vartheta(a_{2k},a_{2k+1}) + \vartheta(b_{2k},b_{2k+1})]$

Where $0 < h = \frac{\alpha}{(1-\beta)} < 1. similarly$

$\vartheta(a_{2k+2},a_{2k+3}) \le \alpha\frac{\vartheta(a_{2k+1},a_{2k+2})}{2(1-\beta)} + \alpha\frac{\vartheta(b_{2k+1},b_{2k+2}}{2(1-\beta)}$

Similarly

$\vartheta(b_{2k+2},b_{2k+3}) \le \alpha\frac{\vartheta(b_{2k+1},b_{2k+2})}{2(1-\beta)} + \alpha\frac{\vartheta(a_{2k+1},a_{2k+2})}{2(1-\beta)}$

Adding above equation, we get

$[\vartheta(a_{2k+2},a_{2k+3}) + \vartheta(b_{2k+2},b_{2k+3})] \le \frac{\alpha}{(1-\beta)}[\vartheta(a_{2k+1},a_{2k+2}) + \vartheta(b_{2k+1},b_{2k+2})]$

$= h[\vartheta(a_{2k+1},a_{2k+2}) + \vartheta(b_{2k+1},b_{2k+2})]$

Continuing in this way,

$(\vartheta(a_n,a_{n+1}) + \vartheta(b_n,b_{n+1})) \le h(\vartheta(a_{n-1},a_n) + \vartheta(b_{n-1},b_n) \le --- \le h^n(\vartheta(a_0,a_1) + \vartheta(b_0,b_1)$

Now, if

$\vartheta(a_n,a_{n+1}) + \vartheta(b_n,b_{n+1}) = \delta_n, then\ \delta_n, then\ \delta_n \le h\,\delta_{n-1} \le --- \le h^n\delta 0$

For $m > n$, we have

$(\vartheta(a_n,a_m) + \vartheta(b_n,b_m)) \le K(\vartheta(a_n,a_{n+1}) +$
$\vartheta bn, bn+1+...+Km-n(\vartheta am-1,am+\vartheta bm-1, b,)$

$\le Kh^n\delta_0 + K^2h^{n+1}\delta 0 + \cdots + K^{m-n}h^{m-1}\delta_0$

$< Kh^n[1 + (Kh) + (Kh)^2 + \cdots]\delta_0$

$= \frac{Kh^n}{1-Kh}\delta_0 \to 0\ as\ n \to \infty$

This shows that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequence in X. Since X is a complete b-metric-like space, there exists $a, b \in X$ such that $a_n \to \infty$ and $b_n \to \infty$.

Step 2 : Now, We show that $a = S(a,b)$ and $b = S(b,a)$.

We suppose on the contrary that $a \ne S(a,b)$ and $b \ne S(b,a)$ so that

$\vartheta(a,S(a,b)) = l_1 > 0\ and\ \vartheta(b,S(b,a)) = l_2 > 0$

Consider

$l_1 = \vartheta(a,S(a,b)) \le K[\vartheta(a,a_{2k+2}) + \vartheta(a_{2k+2},S\,(a,b))]$

$\le K\vartheta(a,a_{2k+2}) + K\vartheta(T(a_{2k+1},b_{2k+1}),S(a,b))$

$\le K\vartheta(a,a_{2k+2}) + Ka\frac{\vartheta(a_{2k+1},a)+\vartheta(b_{2k+1},b)}{2} + K\beta\frac{\vartheta(a,S(a,b))\vartheta(a_{2k+1},T(a_{2k+1},b_{2k+1}))}{1+\vartheta(a_{2k+1},a)+\vartheta(b_{2k+1},b)} +$

$K\gamma\frac{\vartheta(a_{2k+1},S(a,b))\vartheta(a,a_{2k+2})}{1+\vartheta(a_{2k+1},a)+\vartheta(b_{2k+1,b)}}$

By taking $k \to \infty$, we get

$l_1 \le 0$, which is contradiction.

Therefore, $\vartheta(a,S(a,b)) = 0$. this implies $a = S(a,b)$.

Similarly, we can prove that $b = S(b,a)$.

It follows similarly we can show that $a = T(a,b)$ and $T(b,a)$.

So we have proved that (a, b) is a common coupled fixed point of S and T.

Step 3 : We now show that S and T have a unique common coupled fixed point.

Let $(a^*,b^*) \in X \times X$ be another common coupled fixed point of S and T. Then

$\vartheta(a^*,a^*) = \vartheta(S(a,b),T(a^*,b^*))$

$\le \alpha\frac{\vartheta(a,a^*)+\vartheta(b,b^*)}{2} + \beta\frac{\vartheta(a,S(a,b))\vartheta(a^*,T(a^*,T(a^*,b^*))}{1+\vartheta(a,a^*)+\vartheta(b,b^*)} + \gamma\frac{\vartheta(a^*,S(a,b))\vartheta(a,T(a^*,b^*))}{1+\vartheta(a,a^*)+\vartheta(b,b^*)}$

$= \alpha\frac{\vartheta(a,a^*)+\vartheta(b,b^*)}{2} + \beta\frac{\vartheta(a,a)\vartheta(a^*,a^*)}{1+\vartheta(a,a^*)+\vartheta(b,b^*)} + \gamma\frac{\vartheta(a^*,a)\vartheta(a,a^*)}{1+\vartheta(a,a^*)+\vartheta(b,b^*)}$

$\vartheta(a,a^*) \le \frac{\alpha}{2}\vartheta(a,a^*) + \frac{\alpha}{2}\vartheta(b,b^*) + 4\beta\vartheta(a,a^*) + \gamma\vartheta(a^*a)$

$\vartheta(a,a^*) \le \frac{\alpha}{(2-\alpha-8\beta-2\gamma)}\vartheta(b,b^*) \le \frac{\alpha}{(2-\alpha-2\gamma)}\vartheta(b,b^*)$

Similarly, we can easily prove that

$$\vartheta(b,b^*) \leq \frac{\alpha}{(2-\alpha-2\gamma)}\vartheta(a,a^*)$$

Adding, we get

$$\vartheta(a,a^*)+\vartheta(b,b^*) \leq \frac{\alpha}{(2-\alpha-2\gamma)}[\vartheta(a,a^*)+\vartheta(b,b^*)].$$

$$(2-2\alpha-2\gamma)[\vartheta(a,a^*)+\vartheta(b,b^*)] \leq 0$$

$$\vartheta(a,a^*)+\vartheta(b,b^*) = 0.$$

This implies, $a = a^*$ and $b = b^*$

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