Invariant Submanifold of $\tilde{\psi}(p,1)$ Structure Manifold

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Abstract: In this paper, we have studied various properties of a $\tilde{\psi}(p,1)$ structure manifold and its invariant submanifold, where $p$ is odd prime. Under two different assumptions, the nature of induced structure $\psi$, has also been discussed.

Keywords: Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions

1. Introduction

Let $V^m$ be a $C^\infty$ m-dimensional Riemannian manifold imbedded in a $C^\infty$ n-dimensional Riemannian manifold $M^n$, where $m < n$. The imbedding being denoted by $f : V^m \rightarrow M^n$ forms a vector bundle over $f(V)$, which we shall denote by $N(V,M)$. We call $N(V,M)$ the normal bundle of $V^m$. The vector bundle induced by $f$ from $N(V,M)$ is denoted by $N(V,M)$. We denote by $C : N(V) \rightarrow N(V,M)$ the natural isomorphism and by $\eta^i(Y)$ the space of all $C^\infty$ tensor fields of type $(r, s)$ associated with $N(V)$. Thus $\xi^0(V) = \eta^0_0(V)$ is the space of all $C^\infty$ functions defined on $V^m$ while an element of $\eta^1_0(V)$ is a $C^\infty$ vector field normal to $V^m$ and an element of $\xi^1_0(V)$ is a $C^\infty$ vector field tangential to $V^m$.

Let $\tilde{X}$ and $\tilde{Y}$ be vector fields defined along $f(V)$ and $\tilde{X}, \tilde{Y}$ be the local extensions of $X$ and $Y$ respectively. Then $[\tilde{X}, \tilde{Y}]$ is a vector field tangential to $M^n$ and its restriction $[\tilde{X}, \tilde{Y}]/f(V)$ to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $[\tilde{X}, \tilde{Y}]$ is defined as

$$(1.1) \quad [\tilde{X}, \tilde{Y}] = [\tilde{X}, \tilde{Y}]/f(V)$$

Since $B$ is an isomorphism

$$(1.2) \quad [BX, BY] = B [X,Y] \quad \text{for all} \quad X,Y \in \xi^1_0(V)$$

Let $G$ be the Riemannian metric tensor of $M^n$, we define $g$ and $g^*$ on $V^m$ and $N(V)$ respectively as

$$(1.3) \quad g(X_1, X_2) = G(BX_1, BX_2) f, \quad \text{and}$$

$$(1.4) \quad g^*(N_1, N_2) = \tilde{G}(CN_1, CN_2)$$

For all $X_1, X_2 \in \xi^1_0(V)$ and $N_1, N_2 \in \eta^1_0(V)$

It can be verified that $g$ and $g^*$ are the induced metrics on $V^m$ and $N(V)$ respectively.

Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in $M^n$, then $\tilde{\nabla}$ induces a connection $\nabla$ in $f(V)$ defined by

$$(1.5) \quad \nabla_X Y = \tilde{\nabla}_X \tilde{Y}/f(V)$$

where $\tilde{X}$ and $\tilde{Y}$ are arbitrary $C^\infty$ vector fields defined along $f(V)$ and tangential to $f(V)$.

Let us suppose that $M^n$ is a $C^\infty \tilde{\psi}(p,1)$ structure manifold with structure tensor $\tilde{\psi}$ of type (1,1) satisfying

$$(1.6) \quad \tilde{\psi}^p + \tilde{\psi} = 0$$

Let $L$ and $M$ be the complementary distributions corresponding to the projection operators

$$(1.7) \quad L = -\tilde{\psi}^{p-1}, \quad M = I + \tilde{\psi}^p$$

where $I$ denotes the identity operator.
From (1.6) and (1.7), we have
\[ (a) \quad \widehat{I} + \widehat{m} = I \quad (b) \quad \widehat{I}^2 = \widehat{I} \]
\[ (c) \quad \widehat{m}^2 = \widehat{m} \quad (d) \quad \widehat{I} \cdot \widehat{m} = \widehat{m} \cdot \widehat{I} = 0 \]
Let \( D_l \) and \( D_m \) be the subspaces inherited by complementary projection operators \( l \) and \( m \) respectively.

We define
\[ D_l = \{ X \in T_p(V) : lX = X, mX = 0 \} \]
\[ D_m = \{ X \in T_p(V) : mX = X, lX = 0 \} \]
Thus \( T_p(V) = D_l + D_m \)

Also \( \text{Ker} \ l = \{ X : lX = 0 \} = D_m \)
\( \text{Ker} \ m = \{ X : mX = 0 \} = D_l \)
at each point \( p \) of \( f(V) \).

2. Invariant Submanifold of \( \tilde{\psi}(p,1) \) Structure Manifold

We call \( V^m \) to be invariant submanifold of \( M^m \) if the tangent space \( T_p^m(f(V)) \) of \( f(V) \) is invariant by the linear mapping \( \tilde{\psi} \) at each point \( p \) of \( f(V) \). Thus
\[ \tilde{\psi}BX = B\psi X \quad \text{for all} \quad X \in \zeta^1_0(V), \] and \( \psi \) being a (1,1) tensor field in \( V^m \).

Theorem (2.1): Let \( \tilde{N} \) and \( N \) be the Nijenhuis tensors determined by \( \tilde{\psi} \) and \( \psi \) in \( M^m \) and \( V^m \) respectively, then
\[ \tilde{N}(BX, BY) = BN(X,Y) \quad \text{for all} \quad X, Y \in \zeta^1_0(V) \]
Proof: We have, by using (1.2) and (2.1)
\[ (2.3) \quad \tilde{N}(BX, BY) = [\tilde{\psi}BX, \tilde{\psi}BY] + \tilde{\psi}^2[BX, BY] - \tilde{\psi}[BX, \tilde{\psi}BY] \]
\[ = B[\psi X, \psi Y] + B\psi^2[X, Y] - \psi B[\psi X, Y] \]
\[ = B[\psi X, \psi Y] + B\psi^2[X, Y] - \psi B[\psi X, Y] \]
\[ = B[N(X,Y)] \]

3. Distribution \( \tilde{M} \) Never Being Tangential to \( f(V) \)

Theorem (3.1) if the distribution \( \tilde{M} \) is never tangential to \( f(V) \), then
\[ (3.1) \quad \tilde{m}(BX) = 0 \quad \text{for all} \quad X \in \zeta^1_0(V) \]
and the induced structure \( \psi \) on \( V^m \) satisfies
\[ (3.2) \quad \psi^{p-1} = -I \]
Proof: if possible \( \tilde{m}(BX) \neq 0 \). From (2.1) We get
\[ (3.3) \quad \psi^{p-1}BX = B\psi^{p-1}X \quad \text{from (1.7) and (3.3)} \]
\[ \tilde{m}(BX) = (I + \psi^{p-1})BX = BX + B\psi^{p-1}X \]
(3.4) \( \tilde{m}(BX) = B(X + \psi^{p-1}X) \)
This relation shows that \( \tilde{m}(BX) \) is tangential to \( f(V) \) which contradicts the hypothesis. Thus \( \tilde{m}(BX) = 0 \). Using this result in (3.4) and remembering that \( B \) is an isomorphism, We get
\[ (3.5) \quad \psi^{p-1} = -I \] which gives that \( \psi^{(p-1)/2} \) acts as an almost complex structure on \( V^m \). Thus \( V^m \) is even dimensional.

Theorem (3.2) Let \( \tilde{M} \) be never tangential to \( f(V) \), then
\[ (3.6) \quad \tilde{N}(BX, BY) = 0 \]
Proof: We have
\[ (3.7) \quad \tilde{N}(BX, BY) = [\tilde{m}BX, \tilde{m}BY] + \tilde{m}^2[BX, BY] \]
\[ - \tilde{m}[\tilde{m}BX, BY] - \tilde{m}[BX, \tilde{m}BY] \]
Using (1.2), (1.8) (c) and (3.1), we get (3.6).

Theorem (3.3) Let \( \tilde{M} \) be never tangential to \( f(V) \), then
\[ (3.8) \quad \tilde{N}(BX, BY) = 0 \]
Proof: We have
\[ N(BX, BY) = \tilde{I} BX, \tilde{I} BY + \tilde{I}^2 [BX, BY] \]

Using (1.2), (1.8) (a), (b) and (3.1) in (3.9); we get (3.8)

Theorem (3.4) Let \( \mathcal{M} \) be never tangential to \( f(V) \).
Define
\[ \tilde{H}(\tilde{X}, \tilde{Y}) = N(\tilde{X}, \tilde{Y}) - N(\tilde{mX}, \tilde{mY}) \]

For all \( \tilde{X}, \tilde{Y} \in \zeta_0(M) \), then
\[ \tilde{H}(\tilde{X}, \tilde{Y}) = BN(X, Y) \]

Proof: Using \( \tilde{X} = BX, \tilde{Y} = BY \) and (2.2), (3.1) in (3.10) We get (3.11).

4. Distribution \( \mathcal{M} \) Always Being Tangential to \( f(V) \)

Theorem (4.1) Let \( \mathcal{M} \) be always tangential to \( f(V) \), then
\[ \tilde{m}(BX) = BmX \quad (a) \quad \tilde{I}(BX) = BI X \quad (b) \]

Proof: from (3.4), We get (4.1) (a). Also
\[ lX = -\psi^{-1} X \]
\[ lI = -\psi^{-1} X \]
\[ BI = -B \psi^{-1} X \]
Using (2.1) in (4.3)
\[ BI = -\tilde{\psi}^{-1} BX = \tilde{I}(BX) \]

Theorem (4.2) Let \( \mathcal{M} \) be always tangential to \( f(V) \), then \( \tilde{I} \) and \( m \) satisfy
\[ l + m = l \quad (a) \quad \tilde{l}l = 0 \quad (b) \quad \tilde{l}m = m \quad (c) \quad l^2 = l \quad (d) \quad m^2 = m \]

Proof: Using (1.8) and (4.1) We get the results.

Theorem (4.3) If \( \mathcal{M} \) is always tangential to \( f(V) \), then
\[ \psi^{-1} + \psi = 0 \]

Proof: From (2.1)
\[ \tilde{\psi} BX = B \psi X \]

- Using (1.6) in (4.7)
\[ -\tilde{\psi} BX = B \psi X \]

Or \( \psi^{-1} + \psi = 0 \) which is (4.6)

Theorem (4.4): If \( \mathcal{M} \) is always tangential to \( f(V) \), then as in (3.10)
\[ \tilde{H}(BX, BY) = BH(X, Y) \]

Proof: from (3.10) we get
\[ \tilde{H}(BX, BY) = N(BX, BY) - \tilde{N}(mBX, mBY) \]
\[ \tilde{N}(BX, mBY) + \tilde{N}(mBX, mBY) \]

Using (4.1) (a) and (2.2) in (4.9) we get (4.8).

References
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