International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2014): 5.611

# Fuzzy Sumudu Transform for Fuzzy n$^{th}$- Order Derivative and Solving Fuzzy Ordinary Differential Equations

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**Abstract:** ***In this paper, we find formulas of fuzzy Sumudu transform for the fuzzy derivatives of orders three and four. Then, we generalize these formulas to the fuzzy nth- order derivative by using strongly generalized H-differentiability concepts. A fuzzy initial value problem FIVP of order four is provided.***

**Keywords:** Fuzzy numbers, generalized H-differentiability**,** Fuzzy Sumudu transform.

## 1. Introduction

One of the most important applications of integral transforms methods is solving the differential equations. For this purpose a new integral transform, which is called Sumudu transform, was introduced by Watugala [1] who used it to solve ordinary differential equations in engineering control problems. Then Sumudu transform was used by many researchers such as Weerakoon [2] for partial derivatives, provided the complex inversion formula in order to solve the differential equations in sciences of engineering and applied physics.

The concept of the fuzzy derivative was first introduced by Chang and Zadeh in [3], it was followed up by Dubois and Prade in [4] and Puri and Ralescu in [5]. In the field of fuzzy integral transforms Allahviranloo and Ahmadi in [6] have proposed the fuzzy Laplace transforms for solving first-order fuzzy differential equations strongly generalized H-differentiability concept. Salahshour and Allahviranloo in [7] have expressed the fuzzy Laplace transform and then some of its well-known properties are investigated. In addition, an existence theorem is given for fuzzy-valued function which possess the fuzzy Laplace transform. The fuzzy Laplace transform for the nth-order derivative is given initially by Mohammad Ali in [8] [ also by Haydar and Mohammad Ali [9] ] in terms of the number of (ii)-differentiable functions under strongly generalized H-differentiability concept. Ahmadi et al. [10], introduced fuzzy Laplace transform formula on the fuzzy nth-order derivative under strongly generalized H-differentiability concept. The strong relation between Laplace transform and Sumudu transform encouraged many authors to fuzziness the Sumudu transform. The first work was introduced by Ahmad and Abdul Rahman in [11], they proposed a novel procedure for solving fuzzy differential equations through fuzzy Sumudu transform, some basic concepts and properties the fuzzy Sumudu transform on first degree derivative under strongly generalized differentiability concept were studied. Sumudu transform was advanced and designed by Khan et al. [12] for the solution of linear differential models with uncertainty and the fuzzy Sumudu transform of second order derivatives under generalized H-differentiability concepts. Abdul Rahman and Ahmad [13] also proposed some results on the properties of the fuzzy Sumudu transform, such as linearity, preserving fuzzy derivative, shifting and convolution theorem.

This paper is arranged as follows: Basic concepts are given in Section 2. In Section 3, formulas of fuzzy Sumudu transform for the fuzzy derivatives of orders three, four and $n$-th order derivative are found. In Section 4, an example of the fourth order FIVP is solved. In Section 5, conclusions are drawn.

## 2. Basic Concepts

In this section, some necessary definitions and concepts are introduced:

**Definition 1 [7]** A fuzzy number $u$ in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$, $0\le\alpha\le 1$ which satisfy the following requirements:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(\alpha)\le\overline{u}(\alpha),\ 0\le\alpha\le 1$.

**Definition 2. [7]** Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then $z$ is called the H-difference of $x$ and $y$, and it is denoted by $x \ominus y$**.** In this paper, the sign "$\ominus$" always stands for H-difference, and also note that $x \ominus y \neq x + (-1)y$.

**Definition 3 [7]** Let $f:(a,b)\to E$ and $x_0 \in (a,b)$. We say that $f$ is strongly generalized differential at $x_0$ if there exists an element $f'(x_0)\in E$, such that

i. For all $h>0$ sufficiently small, $\exists f\left(x_0+h\right) \ominus f\left(x_0\right)$, $\exists f\left(x_0\right)\ominus f\left(x_0-h\right)$ and the limits (in the metric d)

$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h]=f'(x_0)$ or

ii. For all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/(-h)]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/(-h)]=f'(x_0) \text{ or}$$

iii. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0)$, $\exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/(-h)]=f'(x_0) \text{ or}$$

iv. For all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h)$, $\exists f(x_0)\ominus f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/(-h)]=\lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h]=f'(x_0)$$

**Definition 4 [13]** Let $f:R\to F(R)$ be a continuous fuzzy-valued function. Suppose that $f(ut)\,e^{-t}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty f(ut)\,e^{-t}dt$, is called the fuzzy Sumudu transform and is denoted by:

$$G(u)=S[f(t)]=\int_0^\infty f(ut)\cdot e^{-t}dt,\quad (u\in[-\tau,\tau]),$$

where the variable $u$ is used to factor the variable $t$ in the argument of the fuzzy-valued function. We have:

$$\int_0^\infty f(ut)e^{-t}dt=[\int_0^\infty \underline{f}(ut,\alpha)e^{-t}dt,\ \int_0^\infty \overline{f}(ut,\alpha)e^{-t}dt].$$

From the classical Sumudu transform, we have:

$$s[\underline{f}(t,\alpha)]=\int_0^\infty \underline{f}(ut,\alpha)\cdot e^{-t}dt,$$

and

$$s[\overline{f}(t,\alpha)]=\int_0^\infty \overline{f}(ut,\alpha)\cdot e^{-t}dt.$$

Finally, we have:

$$S[f(t)]=[s[\underline{f}(t,\alpha)],s[\overline{f}(t,\alpha)]].$$

**Theorem 1 [13]**. Let $f:R\to F(R)$ be a continuous fuzzy-valued function and $f$ the primitive of $f'$ on $[0,\infty)$. Then:

$$S[f'(t)]=\frac{G(u)}{u}\ominus\frac{f(0)}{u} \quad \text{where } f \text{ is (i)- differentiable.}$$

or:

$$S[f'(t)]=-\frac{f(0)}{u}\ominus\frac{-G(u)}{u} \quad \text{where } f \text{ is (ii)-differentiable.}$$

**Theorem 2. [12]** Suppose that $f$ and $f'$ are continuous fuzzy-valued functions on $[0,\infty)$ and that $f''$ is piecewise continuous fuzzy-valued function on $[0,\infty)$, then:

If $f$ and $f'$ are (i)-differentiable, then

$$S[f''(t)]=\frac{S[f(t)]}{u^2}\ominus\frac{f(0)}{u^2}\ominus\frac{f'(0)}{u}$$

If $f$ is (i)-differentiable and $f'$ is (ii)-differentiable, then

$$S[f''(t)]=-\frac{f(0)}{u^2}\ominus\left[-\frac{S[f(t)]}{u^2}\right]-\frac{f'(0)}{u}$$

If $f'$ is (i)-differentiable and $f$ is (ii)-differentiable, then

$$S[f''(t)]=-\frac{f(0)}{u^2}\ominus\left[-\frac{S[f(t)]}{u^2}\right]\ominus\frac{f'(0)}{u}$$

If $f$ and $f'$ are (ii)-differentiable, then

$$S[f''(t)]=\frac{S[f(t)]}{u^2}\ominus\frac{f(0)}{u^2}-\frac{f'(0)}{u^2}$$

**Definition 5** [6] Let $f(t)$ be a continuous fuzzy-valued function. Suppose that $f(t)\,e^{-pt}$ is improper fuzzy Riemann-integrable on $[0,\infty)$, then $\int_0^\infty f(t)\,e^{-pt}dt$ is called the fuzzy Laplace transform and is denoted as:

$$L[f(t)]=\int_0^\infty f(t)\cdot e^{-pt}dt,\quad (p>0).$$

**Theorem 3.** [13] Let $f(t)$ be a continuous fuzzy-valued function. If $F$ is the fuzzy Laplace transform of $f(t)$ and $G$ is the fuzzy Sumudu transform of $f(t)$, then:

$$G(u)=\frac{F(1/u)}{u}. \tag{1}$$

**Theorem 4.**[8,9] Suppose that $f(t),f'(t),\dots,f^{(n-1)}(t)$ are differentiable fuzzy valued functions such that $f^{(i_1)}(t),f^{(i_2)}(t),\dots,f^{(i_m)}(t)$ are (ii)-differentiable functions for $0\le i_1<i_2<\dots<i_m\le n-1,\ 0\le m\le n$ and $f^{(p)}(t)$ is (i)-differentiable for $p\ne i_j,\ j=1,2,\dots,m$, and if $\alpha$-cut representation of fuzzy- valued function $f(t)$ is denoted by $f(t)=[\underline{f}(t,\alpha),\overline{f}(t,\alpha)]$., then:

(a) If $m$ is an even number then $f^{(n)}(t)=[\underline{f}^{(n)}(t,\alpha),\overline{f}^{(n)}(t,\alpha)]$.

(b) If $m$ is an odd number then $f^{(n)}(t)=[\overline{f}^{(n)}(t,\alpha),\underline{f}^{(n)}(t,\alpha)]$.

**Theorem 5 [8,9].** Suppose that $f(t),f'(t),\dots,f^{(n-1)}(t)$ be continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f^{(n)}(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$. Let $f^{(i_1)}(t),f^{(i_2)}(t),\dots,f^{(i_m)}(t)$ be (ii)-differentiable functions for $0\le i_1<i_2<\dots<i_m\le n-1$ and $f^{(p)}$ be

(i)-differentiable function for $p \neq i_j, j = 1,2,\ldots,m$ and $f(t) = (\underline{f}(t,\alpha), \overline{f}(t,\alpha))$; then

(1) If $m$ is an even number, we have

$$L(f^{(n)}(t)) = p^n L(f(t)) \ominus p^{n-1} f(0) \otimes \sum_{k=1}^{n-1} p^{n-(k+1)} f^{(k)}(0), \qquad (2)$$

such that

$$\otimes = \begin{cases} \ominus, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an even number} \\ -, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an odd number} \end{cases}$$

(2) If $m$ is an odd number, we have

$$L(f^{(n)}(t)) = -p^{n-1} f(0) \ominus (-p^n) L(f(t)) \otimes \sum_{k=1}^{n-1} p^{n-(k+1)} f^{(k)}(0), \qquad (3)$$

such that

$$\otimes = \begin{cases} \ominus, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided } i < k \textit{ is an odd} \text{ number} \\ -, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided } i < k \textit{ is an even} \text{ number} \end{cases}$$

## 3. Fuzzy Sumudu Transform for Fuzzy nth-Order Derivative

First, we shall find Fuzzy Sumudu Transform for Fuzzy Derivatives of orders three and four in the following two theorems:

**Theorem 6.** Suppose that $f, f'$ and $f''$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f'''$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $f(t) = [\underline{f}(t,\alpha), (\overline{f}(t,\alpha)]$, then:

If $f, f'$ and $f''$ are (i)-differentiable, then

$$S[f'''(t)] = \frac{S[f(t)]}{u^3} \ominus \frac{f(0)}{u^3} \ominus \frac{f'(0)}{u^2} \ominus \frac{f''(0)}{u} \qquad (4)$$

If $f'$ and $f''$ are (i)-differentiable and $f$ is (ii)-differentiable, then

$$S[f'''(t)] = -\frac{f(0)}{u^3} \ominus [-\frac{S[f(t)]}{u^3}] \ominus \frac{f'(0)}{u^2} \ominus \frac{f''(0)}{u} \qquad (5)$$

If $f$ and $f''$ are (i)-differentiable and $f'$ is (ii)-differentiable, then

$$S[f'''(t)] = -\frac{f(0)}{u^3} \ominus [-\frac{S[f(t)]}{u^3}] - \frac{f'(0)}{u^2} \ominus \frac{f''(0)}{u} \qquad (6)$$

If $f$ and $f'$ are (i)-differentiable and $f''$ is (ii)-differentiable, then

$$S[f'''(t)] = -\frac{f(0)}{u^3} \ominus [-\frac{S[f(t)]}{u^3}] - \frac{f'(0)}{u^2} - \frac{f''(0)}{u} \qquad (7)$$

If $f''$ is (i)-differentiable and $f$ and $f'$ are (ii)-differentiable, then

$$S[f'''(t)] = \frac{S[f(t)]}{u^3} \ominus \frac{f(0)}{u^3} - \frac{f'(0)}{u^2} \ominus \frac{f''(0)}{u} \qquad (8)$$

If $f'$ is (i)-differentiable and $f$ and $f''$ are (ii)-differentiable, then

$$S[f'''(t)] = \frac{S[f(t)]}{u^3} \ominus \frac{f(0)}{u^3} - \frac{f'(0)}{u^2} - \frac{f''(0)}{u} \qquad (9)$$

If $f$ is (i)-differentiable and $f'$ and $f''$ are (ii)-differentiable, then

$$S[f'''(t)] = \frac{S[f(t)]}{u^3} \ominus \frac{f(0)}{u^3} \ominus \frac{f'(0)}{u^2} - \frac{f''(0)}{u} \qquad (10)$$

If $f, f'$ and $f''$ are (ii)-differentiable, then

$$S[f'''(t)] = -\frac{f(0)}{u^3} \ominus [-\frac{S[f(t)]}{u^3}] \ominus \frac{f'(0)}{u^2} - \frac{f''(0)}{u} \qquad (11)$$

**Proof**: We prove (6) as follows: Since $f$ and $f''$ are (i)-differentiable and $f'$ is (ii)-differentiable, then by theorem 4(b) we get:

$$f'''(t) = [\overline{f}'''(t,\alpha), \underline{f}'''(t,\alpha)]$$

Therefore, we get:

$$S[f'''(t)] = S[\overline{f}'''(t,\alpha), \underline{f}'''(t,\alpha)] = \left[s[\overline{f}'''(t,\alpha)], s[\underline{f}'''(t,\alpha)]\right]. \qquad (12)$$

We know from the classical Sumudu transform that:

$$s[\underline{f}'''(t,\alpha)] = \frac{s[\underline{f}(t,\alpha)]}{u^3} - \frac{\underline{f}(0,\alpha)}{u^3} - \frac{\underline{f}'(0,\alpha)}{u^2} - \frac{\underline{f}''(0,\alpha)}{u}$$

$$s[\overline{f}'''(t,\alpha)] = \frac{s[\overline{f}(t,\alpha)]}{u^3} - \frac{\overline{f}(0,\alpha)}{u^3} - \frac{\overline{f}'(0,\alpha)}{u^2} - \frac{\overline{f}''(0,\alpha)}{u},$$

Since $f$ is (i)-differentiable and $f'$ is (ii)-differentiable, then by theorem 4 we get:

$$f'(0) = [\underline{f}'(0,\alpha), \overline{f}'(0,\alpha)],$$

$$f''(0) = [\overline{f}''(0,\alpha), \underline{f}''(0,\alpha)],$$

then, equation (12) becomes:

$$S[f'''(t)] = \left[\frac{s[\overline{f}(t,\alpha)]}{u^3} - \frac{\overline{f}(0,\alpha)}{u^3} - \frac{\overline{f}'(0,\alpha)}{u^2} - \frac{\overline{f}''(0,\alpha)}{u}, \frac{s[\underline{f}(t,\alpha)]}{u^3} - \frac{\underline{f}(0,\alpha)}{u^3} - \frac{\underline{f}'(0,\alpha)}{u^2} - \frac{\underline{f}''(0,\alpha)}{u}\right]$$

$$= -\frac{f(0)}{u^3} \ominus [-\frac{S[f(t)]}{u^3}] - \frac{f'(0)}{u^2} \ominus \frac{f''(0)}{u}.$$

**Theorem 7.** Suppose that $f(t), f'(t), f''(t)$ and $f'''(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f^{(4)}(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $f(t)=[\underline{f}(t,\alpha),(\overline{f}(t,\alpha)]$, then

If $f, f', f''$ and $f'''$ are (i)- differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}\ominus\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{13}$$

If $f', f''$ and $f'''$ are (i)-differentiable and $f$ is (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]\ominus\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{14}$$

If $f, f''$ and $f'''$ are (i)-differentiable and $f'$ is (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]-\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{15}$$

If $f, f'$ and $f'''$ are (i)-differentiable and $f''$ is (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]-\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{16}$$

If $f, f'$ and $f''$ are (i)-differentiable and $f'''$ is (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]-\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{17}$$

If $f''$ and $f'''$ are (i)-differentiable and $f$ and $f'$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}-\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{18}$$

If $f'$ and $f'''$ are (i)-differentiable and $f$ and $f''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}-\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{19}$$

If $f'$ and $f''$ are (i)-differentiable and $f$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}-\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{20}$$

If $f$ and $f'''$ are (i)-differentiable and $f'$ and $f''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}\ominus\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{21}$$

If $f$ and $f''$ are (i)-differentiable and $f'$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}\ominus\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{22}$$

If $f$ and $f'$ are (i)-differentiable and $f''$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=\frac{S[f(t)]}{u^4}\ominus\frac{f(0)}{u^4}\ominus\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{23}$$

If $f'''$ is (i)-differentiable and $f, f'$ and $f''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]\ominus\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}\ominus\frac{f'''(0)}{u} \tag{24}$$

If $f''$ is (i)-differentiable and $f, f'$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]\ominus\frac{f'(0)}{u^3}-\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{25}$$

If $f'$ is (i)-differentiable and $f, f''$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]\ominus\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{26}$$

If $f$ is (i)-differentiable and $f', f''$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)]=-\frac{f(0)}{u^4}\ominus[-\frac{S[f(t)]}{u^4}]-\frac{f'(0)}{u^3}\ominus\frac{f''(0)}{u^2}-\frac{f'''(0)}{u} \tag{27}$$

If $f, f', f''$ and $f'''$ are (ii)-differentiable, then

$$S[f^{(4)}(t)] = \frac{S[f(t)]}{u^4} \Theta \frac{f(0)}{u^4} - \frac{f'(0)}{u^3} \Theta \frac{f''(0)}{u^2} - \frac{f'''(0)}{u} \quad (28)$$

**Proof**: We prove (19) as follows: Since $f'$ and $f'''$ are (i)-differentiable and $f$ and $f''$ are (ii)-differentiable, then by theorem 4(a) we get:

$$f^{(4)}(t) = [\underline{f}^{(4)}(t,\alpha), \overline{f}^{(4)}(t,\alpha)]$$

Therefore, we get:

$$S[f^{(4)}(t)] = S[\underline{f}^{(4)}(t,\alpha), \overline{f}^{(4)}(t,\alpha)] = \left[s[\underline{f}^{(4)}(t,\alpha)], s[\overline{f}^{(4)}(t,\alpha)]\right]. \quad (29)$$

We know from the classical Sumudu transform that:

$$s[\underline{f}^{(4)}(t,\alpha)] = \frac{s[\underline{f}(t,\alpha)]}{u^4} - \frac{\underline{f}(0,\alpha)}{u^4} - \frac{\underline{f}'(0,\alpha)}{u^3} - \frac{\underline{f}''(0,\alpha)}{u^2} - \frac{\underline{f}'''(0,\alpha)}{u},$$

$$s[\overline{f}^{(4)}(t,\alpha)] = \frac{s[\overline{f}(t,\alpha)]}{u^4} - \frac{\overline{f}(0,\alpha)}{u^4} - \frac{\overline{f}'(0,\alpha)}{u^3} - \frac{\overline{f}''(0,\alpha)}{u^2} - \frac{\overline{f}'''(0,\alpha)}{u},$$

Since $f'$ is (i)-differentiable and $f$ and $f''$ are (ii)-differentiable, then by theorem 4 we get:

$$f'(0) = [\overline{f}'(0,\alpha), \underline{f}'(0,\alpha)],$$
$$f''(0) = [\overline{f}''(0,\alpha), \underline{f}''(0,\alpha)],$$
$$f'''(0) = [\underline{f}'''(0,\alpha), \overline{f}'''(0,\alpha)],$$

then, equation (29) becomes:

$$S[f^{(4)}(t)] = \left[\frac{s[\underline{f}(t,\alpha)]}{u^4} - \frac{\underline{f}(0,\alpha)}{u^4} - \frac{\underline{f}'(0,\alpha)}{u^3} - \frac{\underline{f}''(0,\alpha)}{u^2} - \frac{\underline{f}'''(0,\alpha)}{u}, \frac{s[\overline{f}(t,\alpha)]}{u^4} - \frac{\overline{f}(0,\alpha)}{u^4} - \frac{\overline{f}'(0,\alpha)}{u^3} - \frac{\overline{f}''(0,\alpha)}{u^2} - \frac{\overline{f}'''(0,\alpha)}{u}\right]$$
$$= \frac{S[f(t)]}{u^4} \Theta \frac{f(0)}{u^4} - \frac{f'(0)}{u^3} - \frac{f''(0)}{u^2} \Theta \frac{f'''(0)}{u}$$

Now, we shall find a generalization for fuzzy Sumudu transform for $n$-th order derivative in the following theorem:

**Theorem 8.** Suppose that $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $f^{(n)}(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$. Let $f^{(i_1)}(t), f^{(i_2)}(t), \ldots, f^{(i_m)}(t)$ be (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ and $f^{(p)}$ be (i)-differentiable function for $p \ne i_j, j = 1, 2, \ldots, m$ and $f(t) = [\underline{f}(t,\alpha), (\overline{f}(t,\alpha)]$, then

(i)If $m$ is an even number, we have

$$S[f^{(n)}(t)] = \frac{S[f(t)]}{u^n} \Theta \frac{f(0)}{u^n} \otimes \sum_{k=1}^{n-1} \frac{1}{u^{n-k}} f^{(k)}(0), \quad (30)$$

such that

$$\otimes = \begin{cases} \Theta, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an even number} \\ -, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an odd number} \end{cases}$$

(2) If $m$ is an odd number, we have

$$S[f^{(n)}(t)] = -\frac{f(0)}{u^n} \Theta [-\frac{S[f(t)]}{u^n}] \otimes \sum_{k=1}^{n-1} \frac{1}{u^{n-k}} f^{(k)}(0), \quad (31)$$

such that

$$\otimes = \begin{cases} \Theta, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided i} < k \textit{ is an odd } \text{number} \\ -, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided i} < k \textit{ is an even } \text{number} \end{cases}$$

**Proof.** We can proof as in the previous theorems, but we shall prove by using the duality relation between fuzzy Laplace and Sumudu transforms as follows: We suppose that $G(u) = S[f(t)]$, $F(p) = L[f(t)]$, $G_n(u) = S[f^{(n)}(t)]$ and $F_n(p) = L[f^{(n)}(t)]$. From duality relation (1), we have:

$$G_n(u) = S[f^{(n)}(t)] = \frac{F_n(1/u)}{u}. \quad (32)$$

Let $m$ is an even number. Then from theorem 5 when $m$ is an even number, equation (32) becomes:

$$G_n(u) = \frac{1}{u}\left[\frac{1}{u^n} F\left(\frac{1}{u}\right) \Theta (\frac{1}{u})^{n-1} f(0) \otimes \sum_{k=1}^{n-1} (\frac{1}{u})^{n-(k+1)} f^{(k)}(0)\right]$$
$$= \frac{1}{u}\left[\frac{1}{u^{n-1}} \{\frac{1}{u} F\left(\frac{1}{u}\right) \Theta f(0)\}\right] \otimes \frac{1}{u} \sum_{k=1}^{n-1} (\frac{1}{u})^{n-(k+1)} f^{(k)}(0)$$
$$= \frac{1}{u^n}[S[f(t)] \Theta f(0)] \otimes \sum_{k=1}^{n-1} (\frac{1}{u})^{n-k} f^{(k)}(0)$$
$$= \frac{S[f(t)]}{u^n} \Theta \frac{f(0)}{u^n} \otimes \sum_{k=1}^{n-1} \frac{1}{u^{n-k}} f^{(k)}(0).$$

such that

$$\otimes = \begin{cases} \ominus, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an even number} \\ -, \textit{if the number of (ii)} - \textit{differentiable functions} \\ \quad f^{(i)}, \textit{provided } i < k \textit{ is an odd number} \end{cases}$$

Let $m$ is an odd number. Then from theorem 5 when $m$ an odd number , equation (32) becomes:

$$G_n(u) = \frac{1}{u}\Bigg[-\frac{1}{u^{n-1}} f(0) \ominus$$

$$(-\frac{1}{u^n}) F\left(\frac{1}{u}\right) \otimes \sum_{k=1}^{n-1} (\frac{1}{u})^{n-(k+1)} f^{(k)}(0)\Bigg]$$

$$= \frac{1}{u}\left[\frac{1}{u^{n-1}}\{-f(0) \ominus (-\frac{1}{u} F\left(\frac{1}{u}\right))\}\right.$$

$$\otimes \frac{1}{u}\sum_{k=1}^{n-1} (\frac{1}{u})^{n-(k+1)} f^{(k)}(0)$$

$$= \frac{1}{u^n}[-f(0) \ominus (-S[f(t)])] \otimes \sum_{k=1}^{n-1} (\frac{1}{u})^{n-k} f^{(k)}(0)$$

$$= \frac{-f(0)}{u^n} \ominus (\frac{-S[f(t)]}{u^n}) \otimes \sum_{k=1}^{n-1} \frac{1}{u^{n-k}} f^{(k)}(0),$$

such that

$$\otimes = \begin{cases} \ominus, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided i} < k \textit{ is an odd } \text{number} \\ -, \text{if the number of (ii)} - \text{differentiable functions} \\ \quad f^{(i)}, \text{provided i} < k \textit{ is an even } \text{number} \end{cases}$$

## 4. Example

**Example 1.** Consider the following fourth order FIVP:

$$y^{(4)}(t) = y(t),$$
$$y(0) = y'(0) = y''(0) = y'''(0) = (\alpha - 1, 1 - \alpha). \tag{33}$$

We note that:

$$y(t) = (\underline{y}(t,\alpha), \overline{y}(t,\alpha)).$$
$$S[y(t)] = [s[\underline{y}(t,\alpha)], s[\overline{y}(t,\alpha)]].$$

By using fuzzy Sumudu transform method we have:

$$S[y^{(4)}(t)] = S[y(t)]. \tag{34}$$

Now, we shall solve this example for 4 cases as follows:

**Case 1** Let consider $y(t), y'(t), y''(t)$ and $y'''(t)$ be (i)-differentiable. Then by (13) equation (34) becomes:

$$\frac{S[y(t)]}{u^4} \ominus \frac{y(0)}{u^4} \ominus \frac{y'(0)}{u^3} \ominus \frac{y''(0)}{u^2} \ominus \frac{y'''(0)}{u}$$
$$= S[y(t)].$$

Thus :

$$s[\underline{y}(t,\alpha)] = \frac{\alpha - 1}{1 - u},$$
$$s[\overline{y}(t,\alpha)] = \frac{1 - \alpha}{1 - u}.$$

Then, we get the $\alpha$-cut representation of solution as follows:

$$\underline{y}(t,\alpha) = (\alpha - 1)\, e^t,$$
$$\overline{y}(t,\alpha) = (1 - \alpha)\, e^t.$$

This is the same result given by Tapaswini and Chakraverty [14].

**Case 2** Let $y'(t), y''(t)$ and $y'''(t)$ be (i)-differentiable and $y(t)$ be (ii)- differentiable. Then by (14), equation (34) becomes:

$$-\frac{y(0)}{u^4} \ominus [-\frac{S[y(t)]}{u^4}] \ominus \frac{y'(0)}{u^3} \ominus \frac{y''(0)}{u^2} \ominus \frac{y'''(0)}{u}$$
$$= S[y(t)]$$

Thus:

$$-u^4 s[\underline{y}(t,\alpha)] + s[\overline{y}(t,\alpha)] = (\alpha - 1)(u^3 + u^2 + u - 1),$$
$$s[\underline{y}(t,\alpha)] - u^4 s[\overline{y}(t,\alpha)] = (1 - \alpha)(u^3 + u^2 + u - 1). \tag{35}$$

The solution of system (35) is as follows:

$$s[\underline{y}(t,\alpha)] = \frac{(\alpha - 1)(1 - u - u^2 - u^3)}{u^4 + 1}$$

$$s[\overline{y}(t,\alpha)] = \frac{(1 - \alpha)(1 - u - u^2 - u^3)}{u^4 + 1}$$

Now, we have:

$$\underline{y}(t,\alpha) = \frac{\alpha - 1}{2} s^{-1}\left[\frac{(-1-\sqrt{2})u + 1}{u^2 - \sqrt{2}u + 1} + \frac{(-1+\sqrt{2})u + 1}{u^2 + \sqrt{2}u + 1}\right]$$

$$= \frac{\alpha - 1}{2} s^{-1}\left[\frac{1 - \frac{u}{\sqrt{2}}}{(1 - \frac{u}{\sqrt{2}})^2 + \frac{u^2}{2}} + (-1-\sqrt{2})\frac{\frac{u}{\sqrt{2}}}{(1 - \frac{u}{\sqrt{2}})^2 + \frac{u^2}{2}} + \frac{1 + \frac{u}{\sqrt{2}}}{(1 + \frac{u}{\sqrt{2}})^2 + \frac{u^2}{2}}\right.$$

$$\left. + (1-\sqrt{2})\frac{\frac{u}{\sqrt{2}}}{(1 + \frac{u}{\sqrt{2}})^2 + \frac{u^2}{2}}\right]$$

Then, we get the $\alpha$-cut representation of solution as follows:

$$\underline{y}(t,\alpha) = \frac{\alpha - 1}{2}[e^{\frac{t}{\sqrt{2}}} \cos\frac{t}{\sqrt{2}} + (-1-\sqrt{2}) e^{\frac{t}{\sqrt{2}}} \sin\frac{t}{\sqrt{2}} + e^{-\frac{t}{\sqrt{2}}} \cos\frac{t}{\sqrt{2}} + (1-\sqrt{2}) e^{-\frac{t}{\sqrt{2}}} \sin\frac{t}{\sqrt{2}}]$$

$$= (\alpha - 1)(\cos\frac{t}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} - \sqrt{2}\sin\frac{t}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} - \sin\frac{t}{\sqrt{2}} \sinh\frac{t}{\sqrt{2}}),$$

$$\overline{y}(t,\alpha) = (1 - \alpha)(\cos\frac{t}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} - \sqrt{2}\sin\frac{t}{\sqrt{2}} \cosh\frac{t}{\sqrt{2}} - \sin\frac{t}{\sqrt{2}} \sinh\frac{t}{\sqrt{2}})$$

**Case 3** Let $y''(t)$ and $y'''(t)$ be (i)-differentiable and $y(t)$ and $y'(t)$ be (ii)-differentiable. Then by (18), equation (34) becomes:

$$\frac{S[y(t)]}{u^4} \ominus \frac{y(0)}{u^4} - \frac{y'(0)}{u^3} \ominus \frac{y''(0)}{u^2} \ominus \frac{y'''(0)}{u}$$
$$= S[y(t)].$$

Thus:

$$s[\underline{y}(t,\alpha)] = \frac{(\alpha-1)(1-u+u^2+u^3)}{1-u^4}$$
$$= (\alpha-1)[\frac{1}{1-u^2} - \frac{u}{1+u^2}]$$
$$s[\overline{y}(t,\alpha)] = (1-\alpha)[\frac{1}{1-u^2} - \frac{u}{1+u^2}]$$

Then, we get the $\alpha$-cut representation of solution is as follows:

$$\underline{y}(t,\alpha) = (\alpha-1)(\cosh t - \sin t),$$
$$\overline{y}(t,\alpha) = (1-\alpha)(\cosh t - \sin t).$$

**Case 4** Let $y(t), y'(t), y''(t)$ and $y'''(t)$ be (ii)-differentiable. Then by (28), equation (34) becomes:

$$\frac{S[y(t)]}{u^4} \ominus \frac{y(0)}{u^4} - \frac{y'(0)}{u^3} \ominus \frac{y''(0)}{u^2} - \frac{y'''(0)}{u}$$
$$= S[y(t)]$$

Thus

$$s[\underline{y}(t,\alpha)] = \frac{\alpha-1}{1+u},$$
$$s[\overline{y}(t,\alpha)] = \frac{1-\alpha}{1+u}.$$

Then, we get the $\alpha$-cut representation of solution as follows:

$$\underline{y}(t,\alpha) = (\alpha-1)\, e^{-t},$$
$$\overline{y}(t,\alpha) = (1-\alpha)\, e^{-t}.$$

## 5. Conclusions

The formulas of fuzzy Sumudu transforms for fuzzy derivatives of orders three and four and fuzzy Sumudu transform for fuzzy derivatives of any order $n$, $n \in Z^+$ are found under strongly generalized H-differentiability. Solutions to FIVP of the fourth order are provided.

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