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Farey to Cantor

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Abstract: ***The Farey fractions lie in [0,1]. Similarly the Cantor middle-$\frac{1}{3}$ set lie in [0,1]. Here we try to construct the Cantor middle-$\frac{1}{3}$ set from Farey sequence.***

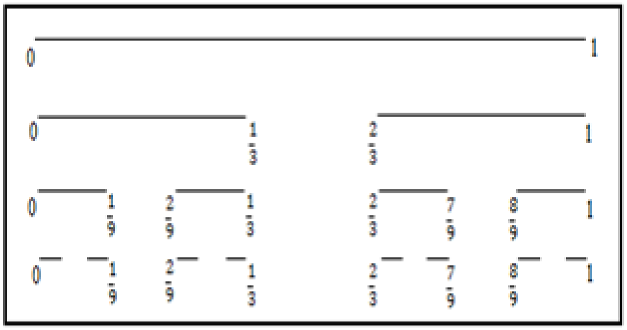
Keywords: Farey Sequence, Non-Reducible Farey Sequence, Non Reducible Farey N-Subsequence, Cantor Sequence.

1. Introduction[2,4,5]

The Farey Sequene is a pattern that has its origin in quite common numbers. The Farey fractions can be found in all sorts of different applications. The Farey sequence was so named for British born geologist, John Farey (1766-1826). In 1816 Farey wrote about the "curious nature of vulgar fractions" in the publication Philosophical Magazine. Given a sequence (F_N) are made up of fractions in lowest terms where the denominator is less than or equal a number N. When the fractions of F_1 are added together from the mediant property, $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, a new fractions falls between the original two is generated. This fraction is called the mediant. The next series is found by adding the first two fractions of F_2 to find the mediant $\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}$. One finds the mediant of the last two fractions in F_2, $\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$, and the next Farey sequence is found. This procedure of finding the mediant between each pair of fraction in the previous Farey sequence is repeated to find the next sequence.

2. Cantor Middle-$\frac{1}{3}$ set:[1]

To construct this set (denoted by C_3), we begin with the interval [0,1] and remove the open set $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the closed interval [0,1]. The set of points that remain after the first step will be called K_1. That is, $K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. In the second step, remove the middle thirds of the two segments of K_1, that is, remove $\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ and set $K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ be what remains after the first two steps. Delete the middle thirds of the four remaining segments of K_2 to get K_3. Repeating this process, the limiting set $C_3 = K_\infty$ is called the Cantor middle-1/3 set. It is presented below.



Thus the cantor set is produced by the iterated process of removing the middle third from the previous segments.

2.1. Farey Sequence

The sequence of all reduced fractions with denominators not exceeding N listed in order of their size is called the Farey sequence of order N.

The Farey Sequence of order 5 is
$$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$

2.2. Non Reducible Farey Sequence

The Sequence of non-reduced fractions with denominators not exceeding N listed in order of their size is called Non Reducible Farey Sequence of order N. The Non Reducible Farey Sequence of order 6 is

$$\widetilde{F_6} = \left\{\frac{0}{1} = \frac{0}{6}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{6}, \frac{2}{5}, \frac{3}{6}, \frac{3}{5}, \frac{4}{6}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{6} = \frac{1}{1}\right\}$$

2.3. Non Reducible Farey N-Subsequence

The sequence of non-reduced Farey fractions with denominators equal to the order of the size N is called Non Reducible Farey N-Subsequence.

The Non Reducible Farey N-Subsequence of order 6 is
$$\tilde{F}_6 = \left\{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\right\}$$

2.4. Complement of a Sequence

The numerator of each element of a sequence subtracted from its order is called the complement of a sequence.

2.5 Theorem

Transform the Non Reducible Farey N-Subsequence of order $N = 3^n$ to Cantor sequence and convert into Cantor set interval, where $n \geq 3$.

Proof:
We prove the theorem by iterative method. Let the Non Reducible Farey N-Subsequence be

$$\widetilde{F_N} = \left\{\left\{\frac{3^{n-1}+1}{3^n}, \frac{3^{n-1}+2}{3^n}, \dots, \frac{2.3^{n-1}-1}{3^n}, \frac{2.3^{n-1}}{3^n}, \frac{2.3^{n-1}+1}{3^n}, \dots, \frac{2.3^{n-1}+(3^{n-1}-1)}{3^n}\right\} \cup \{0,1\}\right\}$$

where $N = 3^n$ is order of the sequence and n denotes the iteration of the sequence. To convert the Farey subsequence into Cantor sequence we remove certain sets from Non Reducible Farey N-Subsequence as given below.

Construction of removed sets $\{\psi_n\}, \{\psi'_n\}$.
Take

$$\{\psi_n\} = \left\{\frac{3^{n-1}+1}{3^n}, \frac{3^{n-1}+2}{3^n}, \dots, \frac{3^{n-1}+(3^{n-1}-1)}{3^n}\right\},$$

$$\{\psi'_n\} = \{N - 3^{n-1} + 1, N - 3^{n-1} + 2, \dots, N - 3^{n-1} + (3^{n-1} - 1)\}$$

When $n = k \geq 3$,

$$\widetilde{F_{3^k}} = \left\{\left\{\frac{3^{k-1}+1}{3^k}, \frac{3^{k-1}+2}{3^k}, \dots, \frac{2.3^{k-1}-1}{3^k}, \frac{2.3^{k-1}}{3^k}, \frac{2.3^{k-1}+1}{3^k}, \dots, \frac{2.3^{k-1}+(3^{k-1}-1)}{3^k}\right\} \cup \{0,1\}\right\}$$

where $N = 3^k$

$$C_{3^k} = \left\{\widetilde{F_{3^k}} / \cup_{i=2}^{k} \frac{1}{3^{k-i}}\{\psi_i\} \cup_{i=2}^{k-1} \frac{1}{3^k}\{\psi'_i\}\right\},$$

Where $\{\psi'_i\} = \{3^k - numerator\{\psi_i\}\}$

The set $C_{3^{k+1}}$ contains the fractions of the set C_{3^k}

$$C_{3^{k+1}} = \left\{\widetilde{F_{3^{k+1}}} / \bigcup_{i=2}^{k+1} \frac{1}{3^{k+1-i}}\{\psi_i\} \bigcup_{i=2}^{k} \frac{1}{3^{k+1}}\{\psi'_i\}\right\}$$

Where

$$\widetilde{F_{3^{k+1}}} = \left\{\frac{1}{3}\{\widetilde{F_{3^k}}\} \cup \left\{\frac{3^k+1}{3^{k+1}}, \frac{3^k+2}{3^{k+1}}, \dots, \frac{2.3^k-1}{3^{k+1}}, \frac{2.3^k}{3^{k+1}}, \frac{2.3^k+1}{3^{k+1}}, \dots, \frac{2.3^k+(3^k-1)}{3^{k+1}}\right\}\right\}$$

$$\cup_{i=2}^{k+1} \frac{1}{3^{k+1-i}}\{\psi_i\} = 3\left(\cup_{i=2}^{k} \frac{1}{3^{k-i}}\{\psi_i\}\right) \cup \{\psi_{k+1}\},$$

$$\cup_{i=2}^{k} \frac{1}{3^{k+1}}\{\psi'_i\} = \frac{1}{3}\left(\cup_{i=2}^{k-1} \frac{1}{3^k}\{\psi'_i\}\right) \cup \frac{1}{3^{k+1}}\{\psi'_k\}$$

Pairing the set of fractions into closed intervals to get Cantor set of intervals.

2.6 Illustration

Transformation the Farey subsequence of order $81 = 3^4$ into Cantor set interval.

Solution:

$$\widetilde{F_{81}} = \left\{0, \frac{1}{81}, \frac{2}{81}, \dots, \frac{80}{81}, 1\right\},$$

$$C_{3^n} = \left\{\widetilde{F_N} / \bigcup_{i=2}^{n} \frac{1}{3^{n-i}}\{\psi_i\} \bigcup_{i=2}^{n-1} \frac{1}{N}\{\psi'_i\}\right\}$$

$$C_{3^4} = \left\{\widetilde{F_{81}} / \bigcup_{i=2}^{4} \frac{1}{3^{4-i}}\{\psi_i\} \bigcup_{i=2}^{3} \frac{1}{81}\{\psi'_i\}\right\}$$

$$C_{3^4} = \left\{\widetilde{F_{81}} / \left\{\frac{1}{3^2}\{\psi_2\} \cup \frac{1}{3}\{\psi_3\} \cup \{\psi_4\} \cup \frac{1}{81}\{\psi'_3\} \cup \frac{1}{81}\{\psi'_2\}\right\}\right\}$$

$$C_{3^4} = \left\{0, \frac{1}{81}, \frac{2}{81}, \frac{3}{81}, \frac{6}{81}, \frac{7}{81}, \frac{8}{81}, \frac{9}{81}, \frac{18}{81}, \frac{19}{81}, \frac{20}{81}, \frac{21}{81}, \frac{24}{81}, \frac{25}{81}, \frac{26}{81}, \frac{27}{81}, \frac{54}{81}, \frac{55}{81}, \frac{56}{81}, \frac{57}{81}, \frac{60}{81}, \frac{61}{81}, \frac{62}{81}, \frac{63}{81}, \frac{72}{81}, \frac{73}{81}, \frac{74}{81}, \frac{75}{81}, \frac{78}{81}, \frac{79}{81}, \frac{80}{81}, 1\right\}$$

Now pairing the consecutive fractions into an interval we get Cantor set interval.

The Cantor Set interval of order 81 is

$$\left[0, \frac{1}{81}\right], \left[\frac{2}{81}, \frac{3}{81}\right], \left[\frac{6}{81}, \frac{7}{81}\right], \left[\frac{8}{81}, \frac{9}{81}\right], \left[\frac{18}{81}, \frac{19}{81}\right], \left[\frac{20}{81}, \frac{21}{81}\right], \left[\frac{24}{81}, \frac{25}{81}\right], \left[\frac{26}{81}, \frac{27}{81}\right], \left[\frac{54}{81}, \frac{55}{81}\right], \left[\frac{56}{81}, \frac{57}{81}\right], \left[\frac{60}{81}, \frac{61}{81}\right], \left[\frac{62}{81}, \frac{63}{81}\right], \left[\frac{72}{81}, \frac{73}{81}\right], \left[\frac{74}{81}, \frac{75}{81}\right], \left[\frac{78}{81}, \frac{79}{81}\right], \left[\frac{80}{81}, 1\right].$$

3. Conclusion

Here we are constructing the Cantor Middle third Sets from Non reduced Farey N-Subsequence. This can be extended to generalized Cantor Middle-ω sets with $0 < \omega < 1$.

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