

Coincidence and Common Fixed Point Theorems in D-Metric Spaces

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Abstract: In this paper we used the concept of compatible mappings of type (P) in D-metric space. Our result generalize the result of Parsai V. and Singh B., Fisher and Pathak.

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1. Introduction

In 1992, a new structure of a generalized metric space was introduced by Dhage on the line of ordinary metric space defined as under:

Let R denoted the real line and X denoted a nonempty set. Let $D : X \times X \times X \rightarrow R$ be a function satisfying properties:

(D₁) $D(x, y, z) \geq 0$ for all $x, y, z \in X$, equality holds if and only if $x = y = z$.

(D₂) $D(x, y, z) = D(x, z, y) = \dots \dots \dots \forall x, y, z \in X$,

(D₃) $D(x, y, z) \leq D(x, y, u) + D(x, u, z) + D(u, y, z) \forall x, y, z, u \in X$,

The function D is called a D-metric for the space X and (X, D) denotes a D-metric space. Generally the usual ordinary metric is called the distance function. D-metric is called diameter function of the points of X (Daghe)

In the last three decades, a number of authors have studied the aspects of fixed point theory in the setting of D-metric spaces. They have been motivated by various concepts already known for metric space and have thus introduced analogous of various concepts in the framework of the D-metric spaces. Khan, Murthy-Chang-Cho-Sharma and Naidu-Prasad introduced the concepts of weakly commuting pairs of self mappings, compatible pairs of self mapping of type (A) in a D-metric space and notion of weak continuity of a D-metric, respectively, and they have proved several common fixed point theorems by using the weakly commuting pairs of self-mappings, compatible pairs of self-mappings of type (A) in a D-metric space and the weak continuity of a D-metric.

In this paper, we use the concept of compatible mappings of type (P) and compare these mappings with compatible mappings and compatible mappings of type (A) in D-metric spaces. In the sequel, we drive some relations between these mappings. Also, we prove a coincidence point a common fixed point theorem for compatible mappings of type (P) in D-metric spaces.

DEFINITIONS [1]: A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be convergent to a point $x \in X$, denoted by $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} D(x_n, x, z) = 0$ for all $z \in X$. The point x is said to be limit of sequence $\{x_n\}$ in X .

DEFINITION [2]: A sequence $\{x_n\}$ in a D-metric space (X, D) is called a Cauchy sequence if $D(x_m, x_n, z) \rightarrow 0$ as $n, m \rightarrow \infty$ for all $z \in X$.

DEFINITION [3]: A D-metric space in which every Cauchy sequence is convergent is called complete.

REMARK [1]: In a D-metric space (X, D) a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the D-metric D is continuous on X .

DEFINITION [4]: Let S and T be mappings from a D-metric space (X, D) into itself. The mappings S and T are said to be compatible if $\lim_{n \rightarrow \infty} D(STx_n, TSx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

DEFINITION [5]: Let S and T be mappings from a D-metric space (X, D) into itself. The mappings S and T are said to be compatible of type (A) if $\lim_{n \rightarrow \infty} D(STx_n, TTx_n, z) = 0$ and $\lim_{n \rightarrow \infty} D(STx_n, SSx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

DEFINITION [6]: Let S and T be mappings from a D-metric space (X, D) into itself. The mappings S and T are said to be compatible of type (P) if $\lim_{n \rightarrow \infty} D(SSx_n, TTx_n, z) = 0$ for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

The following propositions show that Definition [3.5] & [3.6] are equivalent under some conditions:

PROPOSITION [1]: Let S and T be compatible mappings of type(P) from a D-metric space (X, D) into itself. If $St = Tt$ for some t in X , Then $STt = SSt = TTt = TSt$.

PROOF : Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, $n = 1, 2, 3, \dots$ and $St = Tt$. Then we have $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = St$. Since S and T are compatible mappings of type (P), we have $D(SSt, TTt, z) = \lim_{n \rightarrow \infty} D(SSx_n, TTx_n, z) = 0$.

Hence we have $SSt = TTt$. Therefore, $STt = SSt = TTt = TSt$.

Let R^+ denote the set of all non-negative real numbers and F be the family of mappings $\phi : (R^+)^5 \rightarrow R^+$ such that each ϕ is upper-semi-continuous, non-decreasing in each coordinate variable, and for any $t > 0$, $\gamma(t) = \phi(t, t, a_1t, a_2t, t) < t$, where $\gamma : R^+ \rightarrow R^+$ is a mapping with $\gamma(0) = 0$ and $a_1 + a_2 = 3$.

We have prove the following theorems:

THEOREM [1.1]: Let A, B, S and T be mappings from a complete D -metric space (X, D) into itself, satisfying the following conditions:

- [1.1] $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
 - [1.2] $S(X) \cap T(X)$ is a complete subspace of X .
 - [1.3] $[1+p\{D(Ax, Sx, z) + D(By, Ty, z)\}] D(Ax, By, z) \leq p[D^2(Ax, Sx, z) + D^2(By, Ty, z)] + \phi(D(Sx, Ty, z), D(Ax, Sx, z), D(By, Ty, z), D(Ax, Ty, z), D(By, Sx, z))$ for all $x, y, z \in X$, where $\phi \in F$. Then the pairs A, S and B, T have a coincidence point in X .
- For our theorems, we need the following LEMMAS:

LEMMA [1]: For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$, where γ^n denotes the n -times composition of γ .

LEMMA [2] : Let A, B, S and T be mappings from a complete D -metric space (X, D) into itself, satisfying the conditions [3.1.1], [3.4.3]. Then we have the following :

- (a) For every $n \in N_0$, $D(y_n, y_{n+1}, y_{n+2}) = 0$,
- (b) For every $i, j, k \in N_0$, $D(y_i, y_j, y_k) = 0$, where $\{y_n\}$ is the sequence in X defined by [1.4].

PROOF OF THE LEMMA: (a) By(3.1.1) since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for any arbitrary point $x_1 \in X$, there exists a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$[1.4] y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \text{ for } n = 0, 1, 2, \dots$$

In [1.3], taking $x = x_{2n+2}$, $y = x_{2n+1}$, $z = x_{2n}$ we have,
 $[1+p\{D(Ax_{2n+2}, Sx_{2n+2}, y_{2n}) + D(Bx_{2n+1}, Tx_{2n+1}, y_{2n})\}] D(Ax_{2n+2}, Bx_{2n+1}, y_{2n}) \leq p[D^2(Ax_{2n+2}, Sx_{2n+2}, y_{2n}) + D^2(Bx_{2n+1}, Tx_{2n+1}, y_{2n})] + \phi(D(Sx_{2n+2}, Tx_{2n+1}, y_{2n}), D(Ax_{2n+2}, Sx_{2n+2}, y_{2n}), D(Bx_{2n+1}, Tx_{2n+1}, y_{2n}), D(Ax_{2n+2}, Tx_{2n+1}, y_{2n}), D(Bx_{2n+1}, Sx_{2n+2}, y_{2n}))$
 $[1+p\{D(y_{2n+2}, y_{2n+1}, y_{2n}) + D(y_{2n+1}, y_{2n}, y_{2n})\}] D(y_{2n+2}, y_{2n+1}, y_{2n}) \leq p[D^2(y_{2n+2}, y_{2n+1}, y_{2n}) + D^2(y_{2n+1}, y_{2n}, y_{2n})] + \phi(D(y_{2n+1}, y_{2n}, y_{2n}), D(y_{2n+2}, y_{2n+1}, y_{2n}), D(y_{2n+1}, y_{2n}, y_{2n}), D(y_{2n+2}, y_{2n}, y_{2n}), D(y_{2n+1}, y_{2n+1}, y_{2n}))$
 $[1+p\{D(y_{2n+2}, y_{2n+1}, y_{2n}) + 0\}] D(y_{2n+2}, y_{2n+1}, y_{2n})$

$$\leq p[D^2(y_{2n+2}, y_{2n+1}, y_{2n}) + 0] + \phi(0, D(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, 0) D(y_{2n+2}, y_{2n+1}, y_{2n}) \leq \phi(0, D(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, 0) < D(y_{2n+2}, y_{2n+1}, y_{2n}).$$

which is a contradiction. Thus we have $D(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$, similarly, we have $D(y_{2n+1}, y_{2n}, y_{2n-1}) = 0$.

Hence, for $n = 0, 1, 2, \dots$, we have [1.4] $D(y_{n+2}, y_{n+1}, y_n) = 0$.

(b) For all $z \in X$, let $d_n(z) = D(y_n, y_{n+1}, z)$ for $n = 0, 1, 2, \dots$. By (a), we have

$$D(y_n, y_{n+2}, z) \leq D(y_n, y_{n+2}, y_{n+1}) + D(y_n, y_{n+1}, z) + D(y_{n+1}, y_{n+2}, z)$$

$$D(y_n, y_{n+2}, z) \leq D(y_n, y_{n+1}, z) + D(y_{n+1}, y_{n+2}, z)$$

$$D(y_n, y_{n+2}, z) \leq d_n(z) + d_{n+1}(z)$$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in [3.1.3], we have

$$[1+p\{D(Ax_{2n+2}, Sx_{2n+2}, z) + D(Bx_{2n+1}, Tx_{2n+1}, z)\}] D(Ax_{2n+2}, Bx_{2n+1}, z) \leq p[D^2(Ax_{2n+2}, Sx_{2n+2}, z) + D^2(Bx_{2n+1}, Tx_{2n+1}, z)] + \phi(D(Sx_{2n+2}, Tx_{2n+1}, z), D(Ax_{2n+2}, Sx_{2n+2}, z), D(Bx_{2n+1}, Tx_{2n+1}, z), D(Ax_{2n+2}, Tx_{2n+1}, z), D(Bx_{2n+1}, Sx_{2n+2}, z))$$

$$[1+p\{D(y_{2n+2}, y_{2n+1}, z) + D(y_{2n+1}, y_{2n}, z)\}] D(y_{2n+2}, y_{2n+1}, z) \leq p[D^2(y_{2n+2}, y_{2n+1}, z) + D^2(y_{2n+1}, y_{2n}, z)] + \phi(D(y_{2n+1}, y_{2n}, z), D(y_{2n+2}, y_{2n+1}, z), D(y_{2n+1}, y_{2n}, z), D(y_{2n+2}, y_{2n}, z), D(y_{2n+1}, y_{2n+1}, z))$$

$$[1.5] [1+p\{d_{2n+1}(z) + d_{2n}(z)\}] d_{2n+1}(z) \leq p[D^2_{2n+1}(z) + D^2_{2n}(z)] + \phi(d_{2n}(z), d_{2n+1}(z), d_{2n}(z), \{d_{2n}(z) + d_{2n+1}(z)\}, 0)$$

Now, we shall show that $\{d_n(z)\}$ is a non increasing sequence in R^+ . In fact, let $d_{n+1}(z) > d_n(z)$ for some n . By [1.5] we have, $d_{2n+1}(z) < d_{2n+1}(z)$, which is a contradiction in R^+ .

Now, we claim that $d_n(y_m) = 0$ for all non negative integers m, n .

Case 1. $n \geq m$. Then we have $0 = d_m(y_m) \geq d_n(y_m)$.

Case 2. $n < m$. By (M_4) , we have

$$d_n(y_m) \leq d_n(y_{m-1}) + d_{m-1}(y_n) \leq d_n(y_{m-1}) + d_n(y_n) = d_n(y_{m-1})$$

By using the above inequality repeatedly, we have

$$d_n(y_m) \leq d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \dots \leq d_n(y_n) = 0,$$

which completes the proof of our claim.

Finally, let i, j , and k be arbitrary non-negative integers. We may assume that $i < j$. By (M_4) , we have

$$D(y_i, y_j, y_k) \leq d_i(y_j) + d_i(y_k) + D(y_{i+1}, y_j, y_k) = D(y_{i+1}, y_j, y_k).$$

Therefore, by repetition of the above inequality, we have

$$D(y_i, y_j, y_k) \leq D(y_{i+1}, y_j, y_k) \leq \dots \leq D(y_i, y_j, y_k) = 0.$$

This completes the proof.

LEMMA [3]: Let A, B, S and T be mappings from a D -metric space (X, D) into itself satisfying the following conditions [1.1] and [1.3]. Then the sequence $\{y_n\}$ defined by [1.4] is a Cauchy sequence in X .

PROOF OF THE LEMMA : In the proof of LEMMA [2], since $d_n(z)$ is a non increasing sequence in R^+ , by [1.3], we have,

$$[1+p\{D(Ax_2, Sx_2, z) + D(Bx_1, Tx_1, z)\}] D(Ax_2, Bx_1, z) \leq p[d^2(Ax_2, Sx_2, z) + d^2(Bx_1, Tx_1, z)] + \phi(D(Sx_2, Tx_1, z), D(Ax_2, Sx_2, z), D(Bx_1, Tx_1, z), D(Ax_2, Tx_1, z), D(Bx_1, Sx_2, z))$$

$$[1+p\{D(y_2, y_1, z) + D(y_1, y_0, z)\}] D(y_2, y_1, z) \leq p[d^2(y_2, y_1, z) + d^2(y_1, y_0, z)] + \phi(D(y_1, y_0, z), D(y_2, y_1, z), D(y_1, y_0, z), D(y_2, y_0, z), D(y_1, y_1, z))$$

$$[1+p\{d_1(z) + d_0(z)\}] d_1(z) \leq p[d^2_1(z) + d^2_0(z)] + \phi(d_0(z), d_1(z), d_0(z), \{d_0(z) + d_1(z)\}, 0)$$

$$d_1(z) \leq \phi(d_0(z), d_0(z), d_0(z), \{d_0(z) + d_0(z)\}, 0)$$

$d_1(z) \leq \gamma(d_0(z))$
 and $d_2(z) \leq \gamma(d_1(z)) \leq \gamma(\gamma(d_0(z))) = \gamma^2(d_0(z))$.

In general, we have $d_n(z) \leq \gamma^n(d_0(z))$.

Thus, if $d_0(z) > 0$, by LEMMA [3.1] $\lim_{n \rightarrow \infty} d_n(z) = 0$. If $d_0(z) = 0$, we have clearly $\lim_{n \rightarrow \infty} d_n(z) = 0$ since $d_n(z) = 0$ for $n = 1, 2, \dots$

Now, we shall prove that $\{y_n\}$ is a Cauchy sequence in X . Since $\lim_{n \rightarrow \infty} d_n(z) = 0$, it is sufficient to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X . Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in X . Then there exist a point $z \in X$, an $\varepsilon > 0$ and strictly increasing sequences $\{m(k)\}, \{n(k)\}$ of positive integers such that $k \leq n(k) < m(k)$,

$$[1.6] \quad (y_{2n(k)}, y_{2m(k)}, z) \geq \varepsilon \text{ and } D(y_{2n(k)}, y_{2m(k)}, z) < \varepsilon$$

for all $k = 1, 2, \dots$. By LEMMA[3.2] and (M_4) , we have

$$D(y_{2n(k)}, y_{2m(k)}, z) - D(y_{2n(k)}, y_{2m(k-2)}, z) \leq D(y_{2m(k-2)}, y_{2m(k)}, z) \leq d_{2m(k-2)}(z) + d_{2m(k-1)}(z)$$

Since $\{D(y_{2n(k)}, y_{2m(k)}, z) - \varepsilon\}$ and $\{\varepsilon - D(y_{2n(k)}, y_{2m(k-2)}, z)\}$ are sequences in \mathbb{R}^+ and $\lim_{n \rightarrow \infty} d_n(z) = 0$, we have

$$[1.7] \quad \lim_{k \rightarrow \infty} D(y_{2n(k)}, y_{2m(k)}, z) = \varepsilon \text{ and } \lim_{k \rightarrow \infty} D(y_{2n(k)}, y_{2m(k-2)}, z) = \varepsilon$$

Note that, by (M_4) , we have

$$[1.8] \quad |D(x, y, a) - D(x, y, b)| \leq D(a, b, x) + D(a, b, y)$$

for all $x, y, a, b \in X$. Taking $x = y_{2n(k)}, y = a, a = y_{2m(k-1)}$ and $b = y_{2m(k)}$ in [1..8] and using lemma [2] and [1.7], we have

$$[1.9] \quad \lim_{k \rightarrow \infty} D(y_{2n(k)}, y_{2m(k-1)}, z) = \varepsilon.$$

Once again, by using lemma [2], [1..7] and [1.8], we have

$$[1.10] \quad \lim_{k \rightarrow \infty} D(y_{2n(k)+1}, y_{2m(k)}, z) = \varepsilon \text{ and } \lim_{k \rightarrow \infty} D(y_{2n(k-1)}, y_{2m(k-1)}, z) = \varepsilon.$$

Thus, by [1.3], we have,

$$[1.11]$$

$$[1+p\{D(Ax_{2m(k)}, Sx_{2m(k)}, z) + D(Bx_{2n(k+1)}, Tx_{2n(k+1)}, z)\}] D(Ax_{2m(k)}, Bx_{2n(k+1)}, z)$$

$$\leq p[d^2(Ax_{2m(k)}, Sx_{2m(k)}, z) + d^2(Bx_{2n(k+1)}, Tx_{2n(k+1)}, z)] + \phi(D(Sx_{2m(k)}, Tx_{2n(k+1)}, z), D(Ax_{2m(k)}, Sx_{2m(k)}, z), D(Bx_{2n(k+1)}, Tx_{2n(k+1)}, z),$$

$$D(Ax_{2m(k)}, Tx_{2n(k+1)}, z), D(Bx_{2n(k+1)}, Sx_{2m(k)}, z))$$

$$[1+p\{D(y_{2m(k)}, y_{2m(k-1)}, z) + D(y_{2n(k+1)}, y_{2n(k)}, z)\}] D(y_{2m(k)}, y_{2n(k+1)}, z)$$

$$\leq p[d^2(y_{2m(k)}, y_{2m(k-1)}, z) + d^2(y_{2n(k+1)}, y_{2n(k)}, z)] + \phi(D(y_{2m(k-1)}, y_{2n(k)}, z),$$

$$D(y_{2m(k)}, y_{2m(k-1)}, z), D(y_{2n(k+1)}, y_{2n(k)}, z), D(y_{2m(k)}, y_{2n(k)}, z), D(y_{2n(k+1)}, y_{2m(k-1)}, z))$$

As $k \rightarrow \infty$ in [1.11] and noting that d is continuous, we have

$$\varepsilon \leq \phi(\varepsilon, 0, 0, \varepsilon, \varepsilon) < \gamma(\varepsilon) < \varepsilon$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so the sequence $\{y_n\}$ is a Cauchy sequence in X . This completes the proof.

PROOF OF THE THEOREM : By lemma[3], the sequence $\{y_n\}$ defined by [1.2] is a Cauchy sequence in $S(X) \cap T(X)$. Since $S(X) \cap T(X)$ is a complete subspace of X , $\{y_n\}$ converges to a point w in $S(X) \cap T(X)$. On the other hand, since the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ are also Cauchy sequences in $S(X) \cap T(X)$, they also converge to the same limit w . Hence there exist two points u, v in X such that $Su = w$ and $Tv = w$, respectively.

By [1.3], we have

$$[1+p\{D(Au, Su, z) + D(Bx_{2n+1}, Tx_{2n+1}, z)\}] D(Au, Bx_{2n+1}, z)$$

$$\leq p[d^2(Au, Su, z) + d^2(Bx_{2n+1}, Tx_{2n+1}, z)] + \phi(D(Su, Tx_{2n+1}, z), D(Au, Su, z),$$

$$D(Bx_{2n+1}, Tx_{2n+1}, z), D(Au, Tx_{2n+1}, z), D(Bx_{2n+1}, Su, z))$$

$$[1+p\{D(Au, Su, z) + D(y_{2n+1}, y_{2n}, z)\}] D(Au, y_{2n+1}, z)$$

$$\leq p[d^2(Au, Su, z) + d^2(y_{2n+1}, y_{2n}, z)] + \phi(D(Su, y_{2n}, z), D(Au, Su, z), D(y_{2n+1}, y_{2n}, z),$$

$$D(Au, y_{2n}, z), D(y_{2n+1}, Su, z))$$

Since $\lim_{n \rightarrow \infty} d_n(z) = 0$ in the proof of Lemma2, letting $n \rightarrow \infty$, we have

$$[1+p\{D(Au, w, z) + D(w, w, z)\}] D(Au, w, z)$$

$$\leq p[d^2(Au, w, z) + d^2(w, w, z)] + \phi(D(w, w, z), D(Au, w, z), D(w, w, z),$$

$$D(Au, w, z), D(w, w, z))$$

$$D(Au, w, z) \leq \phi(0, D(Au, w, z), 0, D(Au, w, z), 0) < \gamma(D(Au, w, z)) < D(Au, w, z)$$

which is contradiction. Hence $Au = w = Sw$, that is u is a coincidence of A and S .

Similarly, we can show that v is a coincidence point of B and T .

THEOREM [2] : Let A, B, S and T be mappings from a D -metric spaces (X, D) into itself satisfying the conditions (1.1), (3.1.3), (3.1..10) and the following:

[3.2.1] the pairs A, S and B, T are compatible mappings of type (P) .

[3.2.2] the pairs A, S and B, T are sequentially continuous at their coincidence points.

Then A, B, S and T have a unique common fixed point in X .

PROOF OF THEOREM : By theorem [3.1], there exist two points u, v in X such that $Au = Su = w$ and $Bv = Tv = w$, respectively, since A and S are compatible mappings of type (P) , by Proposition 1 $ASu = SSu = SAu = AAu$, which implies that $Aw = Sw$. Similarly B and T are compatible mapping of type (P) we have $Bw = Tw$. Now, we prove that $Aw = w$. If $Aw \neq w$, then by (4.2), we have

$$[1+p\{D(Aw, Sw, z) + D(Bx_{2n+1}, Tx_{2n+1}, z)\}] D(Aw, Bx_{2n+1}, z) \leq p[d^2(Aw, Sw, z) + d^2(Bx_{2n+1}, Tx_{2n+1}, z)] + \phi(D(Sw, Tx_{2n+1}, z), D(Aw, Sw, z),$$

$$D(Bx_{2n+1}, Tx_{2n+1}, z), D(Aw, Tx_{2n+1}, z), D(Bx_{2n+1}, Sw, z))$$

$$[1+p\{D(Aw, Sw, z) + D(y_{2n+1}, y_{2n}, z)\}] D(Aw, y_{2n+1}, z)$$

$$\leq p[d^2(Aw, Sw, z) + d^2(y_{2n+1}, y_{2n}, z)] + \phi(D(Sw, y_{2n}, z), D(Aw, Sw, z),$$

$$D(y_{2n+1}, y_{2n}, z), D(Aw, y_{2n}, z), D(y_{2n+1}, Sw, z))$$

Since $\lim_{n \rightarrow \infty} d_n(z) = 0$ in the proof of Lemma2, letting $n \rightarrow \infty$, we have

$$[1+p\{D(Aw, w, z) + D(w, w, z)\}] D(Aw, w, z)$$

$$\leq p[d^2(Aw, w, z) + d^2(w, w, z)] + \phi(D(w, w, z), D(Aw, w, z), D(w, w, z),$$

$$D(Aw, w, z), D(w, w, z))$$

$$D(Aw, w, z) \leq \phi(0, D(Aw, w, z), 0, D(Aw, w, z), 0) < \gamma(D(Aw, w, z)) < D(Aw, w, z)$$

$$< D(Aw, w, z)$$

which is contradiction.

Hence $Aw = w = Sw$.

Similarly, we have $Bw = Tw = w$.

This means that w is a common fixed point of A, B, S and T . The uniqueness of the fixed point w follows from [1.3].

This complete the proof.

Reference

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