

On The Homogeneous Biquadratic Equation With 5 Unknowns $x^4 - y^4 = 26 (z^2 - w^2) R^2$

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Abstract: The Homogenous biquadratic equation with five unknowns given by $x^4 - y^4 = 26 (z^2 - w^2) R^2$ is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations $x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1$ and employing the method of factorization different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Star number, Pronic number and Gnomonic number are exhibited.

Keywords: Homogeneous equation, Integral solutions, Polygonal number and special number

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Notations used:

$t_{m,n}$ - Polygonal number of rank n with sides m.
 s_n - Star number
 p_n - Pronic number
 G_n - Gnomonic number

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-12]. In this context one may refer [4-10] for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concerns with the homogeneous biquadratic equation with five unknown $x^4 - y^4 = 26 (z^2 - w^2) R^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The biquadratic equation with five unknowns to be solved for its non-zero distinct integral solution is

$$x^4 - y^4 = 26 (z^2 - w^2) R^2 \quad (1)$$

Consider the transformations

$$x = u + v, y = u - v, z = 2uv + 1, w = 2uv - 1 \quad (2)$$

On substituting (2) in (1), we get

$$u^2 + v^2 = 26R^2 \quad (3)$$

2.1 Pattern: I

Assume $26 = (5+i)(5-i)$ (4)
 and $R = a^2 + b^2 = (a+ib)(a-ib)$ (5)

Using (4) and (5) in (3) and employing the method of factorization, we get.

$$(u+iv)(u-iv) = (5+i)(5-i)(a+ib)^2(a-ib)^2$$

On equating the positive and negative factors, we get

$$u+iv = (5+i)(a+ib)^2$$

$$u-iv = (5-i)(a-ib)^2$$

On equating real and imaginary parts, we obtain

$$u = u(a, b) = 5a^2 - 5b^2 - 2ab$$

$$v = v(a, b) = a^2 - b^2 + 10ab$$

On substituting u and v in (2) we get the values of x, y, z and w. The non-zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$$x = x(a, b) = 6a^2 - 6b^2 + 8ab$$

$$y = y(a, b) = 4a^2 - 4b^2 - 12ab$$

$$z = z(a, b) = 2(5a^4 + 5b^4 - 30a^2b^2 + 48a^3b - 48ab^3) + 1$$

$$w = w(a, b) = 2(5a^4 + 5b^4 - 30a^2b^2 + 48a^3b - 48ab^3) - 1$$

$$R = R(a, b) = a^2 + b^2$$

Properties:-

- $x(a, 1) + y(a, 1) - P_n - S_n \equiv 0 \pmod{11}$
- $R(1, 2a) - 4t_{4,a} \equiv 0 \pmod{1}$
- $z(2a, 1) - 3[x(a, 1) + y(a, 1)] + 26t_{4,a} - G_{6a} \equiv 0 \pmod{32}$
- $x(a+1, a+1) - ct_{16,a} + 4G_a = 8$
- $y(a+1, a+1) + t_{26,a} + t_{30,a} + 14t_{4,a} \equiv 0 \pmod{12}$

2.2 Pattern: II

Also 26 can be chosen in equation (3) as

$$26 = (1+5i)(1-5i) \quad (6)$$

Using (5) and (6) in equation (3) it is written in factorizable form as

$$(u + iv)(u - iv) = (1 + 5i)(1 - 5i)(a + ib)^2(a - ib)^2$$

On equating the positive and negative factors, we get,

$$(u + iv) = (1 + 5i)(a + ib)^2$$

$$(u - iv) = (1 - 5i)(a - ib)^2$$

On equating real and imaginary parts, we have

$$u = u(a, b) = a^2 - b^2 - 10ab$$

$$v = v(a, b) = 5a^2 - 5b^2 + 2ab$$

Substituting the values of u and v in (2), the non-zero distinct values of x, y, z, w and R satisfying (1) are given by

$$x = x(a, b) = 6a^2 - 6b^2 - 8ab$$

$$y = y(a, b) = -4a^2 + 4b^2 - 12ab$$

$$z = z(a, b) = 2(5a^4 + 5b^4 - 30a^2b^2 + 48a^3b - 48ab^3) + 1$$

$$w = w(a, b) = 2(5a^4 + 5b^4 - 30a^2b^2 + 48a^3b - 48ab^3) - 1$$

$$R = R(a, b) = a^2 + b^2$$

Properties:-

1. $x(a, 1) + R(a, 1) - 7t_{4,a} - G_{4a} \equiv 0 \pmod{4}$
2. $2y(1, b) - (R(1, 2b))^2 + G_{12b} \equiv 0$
3. $x(n, n-1) - G_{6n} + S_n + 2t_{4,n} - G_n + 3 = 0$
4. $y(2a, 2a) + 48t_{4,a} = 0$
5. $R(a(a+1), 1) - P_a + t_{6,A} - 2t_{4,A} - 2 = 0$

2.3 Pattern: III

Rewrite (3) as

$$1 * u^2 = 26R^2 - v^2 \quad (7)$$

$$\text{Assume } u = 12a^2 - b^2 = (\sqrt{26}a + b)(\sqrt{26}a - b) \quad (8)$$

$$\text{Write 1 as } 1 = (\sqrt{26} + 5)(\sqrt{26} - 5) \quad (9)$$

Using (8) and (9) in (7) it is written in factorizable form as

$$(\sqrt{26} + 5)(\sqrt{26} - 5)(\sqrt{26}a + b)^2(\sqrt{26}a - b)^2 = (\sqrt{26}R + v)(\sqrt{26}R - v) \quad (10)$$

On equating the rational and irrational parts, we get

$$(\sqrt{26} + 5)(\sqrt{26}a + b)^2 = (\sqrt{26}R + v)$$

$$(\sqrt{26} - 5)(\sqrt{26}a - b)^2 = (\sqrt{26}R - v)$$

On equating the real and imaginary parts, we get

$$R = R(a, b) = 26a^2 + b^2 + 10ab$$

$$v = v(a, b) = 130a^2 + 5b^2 + 52ab$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x, y, z, R and w satisfying (1) are given by

$$x = x(a, b) = 156a^2 + 4b^2 + 52ab$$

$$y = y(a, b) = 104a^2 - 6b^2 - 52ab$$

$$z = z(a, b) = 2(3380a^4 - 5b^4 + 1352a^3b - 52ab^3) + 1$$

$$w = w(a, b) = 2(3380a^4 - 5b^4 + 1352a^3b - 52ab^3) - 1$$

$$R = R(a, b) = 26a^2 + b^2 + 10ab$$

Properties:

1. $R(n+1, 1) - ct_{16,n} - 18t_{4,n} - G_{27n} \equiv 0 \pmod{37}$
2. $x(n+1, n+2) + y(n+3, 1) - ct_{16,n} - 308t_{4,n} - G_{266n} \equiv 0 \pmod{208}$

3. $R(1, 2n) - 4P_n - 8G_n \equiv 0 \pmod{27}$
4. $x(2n, 3n) - 1072t_{4,n} = 0$
5. $y(2n, 3n) - S_n - 44t_{4,n} - 3G_n = 0$

2.4 Pattern: 4

Rewrite (3) as

$$1 * v^2 = 26R^2 - u^2 \quad (11)$$

$$\text{Write 1 as } 1 = \frac{(\sqrt{26} - 1)(\sqrt{26} + 1)}{25} \quad (12)$$

$$\text{Assume } v = 26a^2 - b^2 = (\sqrt{26}a + b)(\sqrt{26}a - b) \quad (13)$$

Using (12) and (13) in (11), it is written in factorizable form as

$$\frac{(\sqrt{26} - 1)(\sqrt{26} + 1)}{25} (\sqrt{26}a + b)^2 (\sqrt{26}a - b)^2 = (\sqrt{26}R + u)(\sqrt{26}R - u) \quad (14)$$

On equating the rational and irrational factors, we get

$$R = R(a, b) = \frac{1}{5}(26a^2 + b^2 + 2ab)$$

$$u = u(a, b) = \frac{1}{5}(26a^2 + b^2)52ab \quad (15)$$

Replacing a by 5A and b by 5B in the above equations (13) and (15), we get

$$R = R(A, B) = 130A^2 + 5B^2 + 10AB$$

$$u = u(A, B) = 130A^2 + 5B^2 + 260AB$$

$$v = v(A, B) = 650A^2 - 25B^2$$

On substituting the values of u and v in (2), the non-zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$$x = x(A, B) = 780A^2 - 20B^2 + 260AB$$

$$y = y(A, B) = -520A^2 + 30B^2 + 260AB$$

$$z = z(A, B) = 2(84500A^4 - 125B^4 + 16900A^3B - 6500AB^3) + 1$$

$$w = w(A, B) = 2(84500A^4 - 125B^4 + 10900A^3B - 6500AB^3) - 1$$

$$R = R(A, B) = 130A^2 + 5B^2 + 10AB$$

Properties

1. $R(A, 1) - x(A, 1) + 650t_{4,A} - 135G_A \equiv 0 \pmod{14}$
2. $R(1, 2n) - y(1, 2n) - 20P_n - t_{242,n} - t_{806,n} + 201G_n \equiv 0 \pmod{389}$
3. $x(1, 4n) + t_{642,n} - t_{443,n} + 361G_n \equiv 0 \pmod{781}$
4. $y(1, 3B) + t_{4,B} - P_B - 529G_B - t_{542,B} + 519 = 0$
5. $R(A+1, A) - t_{292,A} - 207G_A \equiv 0 \pmod{131}$

2.5 Pattern: 5

Write (3) as $(u + R)(u - R) = (5R + v)(5R - v)$ (16)

Which is expressed is the form of ratio as

$$\frac{u + R}{5R + v} = \frac{5R - v}{u - R} = \frac{A}{B}, B \neq 0 \quad (17)$$

This is equivalent to the following two equations,

$$-uA + R(5B + A) - VB = 0$$

$$uB + R(B - 5A) - VA = 0$$

On solving the above equations by the method of cross multiplication we get,

$$u = u(A, B) = -A^2 - B^2$$

$$R = R(A, B) = A^2 + B^2$$

$$v = v(A, B) = -5A^2 + 5B^2 + 2AB$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x, y, z, w and R satisfying (1) are given by,

$$x = x(A, B) = -6A^2 + 4B^2 + 2AB$$

$$y = y(A, B) = 4A^2 - 6B^2 - 2AB$$

$$z = z(A, B) = 2[5A^4 - 5B^4 - 2A^3B - 2AB^3] + 1$$

$$w = w(A, B) = 2[5A^4 - 5B^4 - 2A^3B - 2AB^3] - 1$$

$$R = R(A, B) = A^2 + B^2$$

Properties:

1. $x(A, 1) + R(A, 1) + t_{8,A} + 2t_{4,A} \equiv 0 \pmod{5}$
2. $y(n, n+1) - t_{8,n} + 6G_n + P_n \equiv 0 \pmod{2}$
3. $R(2, 2n) - 4t_{4,n} \equiv 0 \pmod{4}$
4. $y(2A, A) - S_n - 3G_A = 0$
5. $R(A+1, A-1) - 2t_{4,A} \equiv 0 \pmod{2}$

3. Conclusion

It is worth to note that in (2), the transformations for z and w may be considered as $z = 2u + v$ and $w = 2u - v$. For this case, the values of x, y and R are the same as above where as the values of z and w changes for every pattern. To conclude one may consider biquadratic equations with multivariables (≥ 5) and search for their non-zero distinct integer solutions along with their corresponding properties.

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