On Ternary Quadratic Diophantine Equation $8x^2 + 8y^2 - 15xy = 40z^2$

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Abstract: Five different methods of the non-zero integral solutions of the ternary quadratic homogeneous Diophantine equation $8x^2 + 8y^2 - 15xy = 40z^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: Ternary Quadratic, homogenous cone, integer solutions, special numbers.

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Notations used

$t_{m,n}$ - Polygonal number of rank n with sides m. $C_{t_{m,n}}$ - Centered Polygonal number of rank n with sides m $s_n$ - Star number $P_n$ - Pronic number $G_n$ - Gnomonic number

1. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting five different methods of the non-zero integral solutions the ternary quadratic homogeneous Diophantine equation $8x^2 + 8y^2 - 15xy = 40z^2$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Method of Analysis

The ternary quadratic homogeneous Diophantine equation to be solved is

$$8x^2 + 8y^2 - 15xy = 40z^2$$

The substitution of linear transformations $x = u + v$ and $y = u - v$ ($u \neq v \neq 0$) in (1) leads to

$$u^2 + 31v^2 = 40z^2$$

The above equation is solved through different methods and different patterns of integer solution to (1) are obtained.

2.1 Pattern: I

Write 40 as

$$40 = (3 + i \sqrt{31})(3 - i \sqrt{31})$$

Assume $z = a^2 + 31b^2$, where $a, b > 0$.

Using (4) and (5) in (3) and applying the method of factorization, define

$$(u + i \sqrt{31}v) = (3 + i \sqrt{31})(a + i \sqrt{31}b)^2$$

Equating the real and imaginary parts, we have

$$u = u(a, b) = 3a^2 - 93b^2 - 62ab$$

$$v = v(a, b) = a^2 - 31b^2 + 6ab$$

Substituting the above $u$ and $v$ in equation (2) the value of $x$ and $y$ are given by

$$x = x(a, b) = 4a^2 - 124b^2 - 56ab$$

$$y = y(a, b) = 2a^2 - 62b^2 - 68ab$$

Thus (5) and (7) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:-

1. $x(a, b) = 3a^2 - 93b^2 - 62ab$
2. $y(a, b) = 2a^2 - 62b^2 - 68ab$
3. $a, b > 0$

2.2 Pattern: II

Consider (3) as $u^2 - 9z^2 = 31(z^2 - v^2)$ (8)

Write (8) in the form of ratio as

$$u + 3z = a(z + v)$$

$$u - 3z = b(z - v)$$

This is equivalent to the following two equations

$$-3u + 3v = a(z + v) - 3b(z - v)$$

$$b(u + 3z) = a(z + v) - 3b(z - v)$$

On employing the method of cross multiplication, we get

$$u = -3a^2 + 93b^2 - 62ab$$

$$v = -a^2 + 31b^2 + 6ab$$

Substituting the values of $u$ and $v$ from (9) in (2) the non-zero distinct integer values of $x, y$ are given by

$$x = x(a, b) = -4a^2 + 124b^2 - 56ab$$

$$y = y(a, b) = -2a^2 + 62b^2 - 68ab$$

Thus (10) and (11) represent the non-zero distinct integer solution of equation (1) in two parameters.
Properties:-
1. \( z (\beta + 3, \beta + 3) + 32 t_4 \beta + G_{90b} \equiv 0 \pmod{287} \)
2. \( y (3, 2\beta) - 245 t_4 \beta + G_{90b} \equiv 0 \pmod{17} \)
3. \( x (\alpha, 1) + 4 t_4 \alpha + G_{23a} \equiv 0 \pmod{125} \)
4. \( x(2, \alpha) + y(2, \alpha) - 186 t_4 a + 124 G_{a} + 23 = 1 \).
5. \( x(3,2\beta) - y(2,3\beta) + I_{150b} + 151 p^b + 151 t_4 b + 28 = 0 \)

Note:-
(8) also be expressed in the form of ratio in three different ways as follows.
(1) \( \frac{u + 3z}{31(z + v)} = \frac{z - v}{u - 3z} = \frac{\alpha}{\beta} \) \( \beta \neq 0 \).
(2) \( \frac{u + 3z}{31(z + v)} = \frac{31(z - v)}{u - 3z} = \frac{\alpha}{\beta} \) \( \beta \neq 0 \).
(3) \( \frac{u + 3z}{31(z - v)} = \frac{z + v}{u - 3z} = \frac{\alpha}{\beta} \) \( \beta \neq 0 \).

Repeating the analysis as above, we get different pattern of solution to (1).

2.3 Pattern: III
Rewrite (3) as \( 31v^2 = 40z^2 - u^2 \) (12)
Write 31 as,
\( 31 = (2 \sqrt{10} + 3)(2 \sqrt{10} - 3) \) (13)
Let \( v = 40a^2 - b^2 \) (14)

Using (13) and (14) in (12) and employing the method of factorization, we write
\( (2 \sqrt{10} z + u) = (2 \sqrt{10} + 3)(2 \sqrt{10} a + b)^2 \)

Equating the rational and irrational parts, we have
\( z = z(a, b) = 40a^2 + b^2 + 6ab \) (15)
\( u = u(a, b) = 120a^2 + 3b^2 + 80ab \) (16)
Substituting (14) & (16) in (2), the values of \( x \) and \( y \) are
\( x = x(a, b) = 160a^2 + 2b^2 + 80ab \)
\( y = y(a, b) = 80a^2 + 4b^2 + 80ab \)
Thus (17) and (15) represent the integer solution to (1).

Properties:
1. \( x(a, 1) - 160 t_4 a - G_{90a} \equiv 0 \pmod{3} \)
2. \( y (3a, 1) - 360 t_4 a - G_{90a} \equiv 0 \pmod{2} \)
3. \( x (a, 2) - z (a, 2) - 160 t_4 a - G_{90a} \equiv 0 \pmod{9} \)
4. \( y(a, 3) - 80 t_4 - G_{120a} \equiv 0 \pmod{37} \)
5. \( y(a, 3) - 164 t_4 a = 0 \)

3. Conclusion
In this paper, we have presented different patterns of integer solutions to the ternary quadratic equation \( 8x^2 + 8y^2 - 15xy = 40z^2 \) representing the cone. As this Diophantine equation is rich in variety, one may attempt to find integer solutions to other choices of equations along with suitable properties.

References

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