

$fagr\omega$ -Connectedness in Fine-Topological Spaces

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Abstract: In this paper, we introduce the notions of $fagr\omega$ -separated sets and $fagr\omega$ -connectedness in fine-topological spaces and study some of their properties in fine-topological spaces.

Keywords: Fine-open sets, $fagr\omega$ -separated sets, $fagr\omega$ -connected sets, $fagr\omega$ -connected space.

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1. Introduction

Connectedness is one of the topological properties that is used to distinguish topological spaces. Many researchers have investigated the basic properties of connectedness. Stronger and weaker forms of connectedness have been introduced and investigated by Noiri [3], Reilly et al. [5] and Benchali and Bansali [1].

In this paper, we introduce the notions of $fagr\omega$ -separated sets, $fagr\omega$ -connectedness and $fagr\omega$ -disconnectedness and also, we study some of their properties in fine-topological spaces

2. Preliminaries

Throughout this paper X and Y denotes topological spaces (X, τ) and (Y, τ) respectively on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $cl(A)$ and $agr\omega-cl(A)$ denote the closure and $agr\omega$ closure of A , respectively. The complement of a set A of (X, τ) is denoted by A^c :

Definition 2.1 A subset A of a topological space (X, τ) is called regular open if $A = int(cl(A))$ (cf. [7]).

Definition 2.2 A subset A of a topological space (X, τ) is called regular semi-open if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$ (cf. [2]).

Definition 2.3 A subset A of a topological space (X, τ) is called $agr\omega$ -closed if $acl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open. A subset A of a topological space (X, τ) is called $agr\omega$ -open if A^c is $agr\omega$ -closed (cf. [6]).

Definition 2.4 A function $f : (X, \tau) \rightarrow (Y, \tau')$ is said to be $agr\omega$ -continuous if $f^{-1}(V)$ is an $agr\omega$ -closed set of (X, τ) for every closed set V of (Y, τ') (cf. [6]).

Definition 2.5 A function $f : (X, \tau) \rightarrow (Y, \tau')$ is said to be $agr\omega$ -irresolute if $f^{-1}(V)$ is an $agr\omega$ -closed set of (X, τ) for every $agr\omega$ -closed set V of (Y, τ') (cf. [7]).

Definition 2.6 Let A be a subset of X . Then $agr\omega$ -closure of A is the intersection of all $agr\omega$ -closed sets containing A (cf. [7]).

Definition 2.7 Let (X, τ) be a topological space. Two non-empty subsets A and B are said to be $agr\omega$ -separated iff $A \cap agr\omega-cl(B) = \phi$ and $agr\omega-cl(A) \cap B = \phi$ i.e., $[A \cap agr\omega-cl(B)] \cup [agr\omega-cl(A) \cap B] = \phi$ (cf. [8]).

Definition 2.8 Let (X, τ) be a topological space and $A \in X$. A point $x \in X$ is said to be an $agr\omega$ -adherent point of A if every $agr\omega$ -open set containing x , contains at least one point of A (cf. [8]).

Definition 2.9 If $X = A \cup B$ such that A and B are non-empty $agr\omega$ -separated sets then A, B form a $agr\omega$ -separation of X (cf. [8]).

Definition 2.10 A topological space (X, τ) is said to be $agr\omega$ -connected if X cannot be written as a union of two disjoint non-empty $agr\omega$ -open sets. If X is not $agr\omega$ -connected then it is $agr\omega$ -disconnected (cf. [8]).

Definition 2.11 A topological space (X, τ) is said to be $Tagr\omega$ -space if every $agr\omega$ -closed set is closed in X (cf. [8]).

Definition 2.12 Let (X, τ) be a topological space we define $\tau(A_\alpha) = \tau_\alpha$ (say) $= \{G_\alpha (\neq X) : G_\alpha \cap A_\alpha \neq \phi, \text{ for } A_\alpha \in \tau \text{ and } A_\alpha \neq \phi, X, \text{ for some } \alpha \in J, \text{ where } J \text{ is the index set.}\}$ Now, we define

$$\tau_f = \{\phi, X, \cup_{\{\alpha \in J\}} \{\tau_\alpha\}\}$$

The above collection τ_f of subsets of X is called the fine collection of subsets of X and (X, τ, τ_f) is said to be the fine space X generated by the topology τ on X (cf. [4]).

Definition 2.13 A subset U of a fine space X is said to be a fine-open set of X , if U belongs to the collection τ_f and the complement of every fine-open sets of X is called the fine-closed sets of X and we denote the collection by F_f (cf. [4]).

Definition 2.14 Let A be a subset of a fine space X , we say that a point $x \in X$ is a fine limit point of A if every fine-open set of X containing x must contains at least one point of A other than x (cf. [4]).

Definition 2.15 Let A be the subset of a fine space X , the fine interior of A is defined as the union of all fine-open sets contained in the set A i.e. the largest fine-open set contained in the set A and is denoted by f_{int} (cf. [4]).

Definition 2.16 Let A be the subset of a fine space X , the fine closure of A is defined as the intersection of all fine-closed sets containing the set A i.e. the smallest fine-closed set containing the set A and is denoted by f_{cl} (cf. [4]).

Definition 2.17 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is called fine-irresolute (or f-irresolute) if $f^{-1}(V)$ is fine-open in X for every fine-open set V of Y (cf. [4]).

Definition 2.18 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be αf -irresolute if $f^{-1}(V)$ is αf -open in X for every αf -open set V of Y (cf. [4]).

Definition 2.19 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be βf -irresolute if $f^{-1}(V)$ is βf -open in X for every βf -open set V of Y (cf. [4]).

Definition 2.20 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be pf -irresolute if $f^{-1}(V)$ is pf -open in X for every pf -open set V of Y (cf. [4]).

Definition 2.21 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be sf -irresolute if $f^{-1}(V)$ is sf -open in X for every sf -open set V of Y (cf. [4]).

3. $fagr\omega$ – Closed Sets

In this section we have defined $fagr\omega$ -closed sets in fine-topological space and studied some of their properties.

Definition 3.1 A subset A of a fine-topological space (X, τ, τ_f) is called fine-regular open if $A = f_{int}(f_{cl}(A))$.

Example 3.1 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{b, c\}\}$, $\tau_f = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $F_f = \{\phi, X, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. It may be easily checked that the only fine-regular open sets of fine-topological space (X, τ, τ_f) are $\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}$.

Definition 3.2 A subset A of a fine-topological space (X, τ, τ_f) is called fine-regular semi-open if there is a fine-regular open set U such that $U \subseteq A \subseteq f_{cl}(U)$.

Example 3.2 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{b, c\}\}$, $\tau_f = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $F_f = \{\phi, X, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}\}$. It may be easily checked that the only fine-regular semi open sets of fine-topological space (X, τ, τ_f) are $\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}$.

Definition 3.3 A subset A of a fine-topological space (X, τ, τ_f) is called $fagr\omega$ -closed if $f_{acl}(A) \subseteq U$ whenever $A \subseteq U$ and U is fine-regular semi-open. A subset A of a fine-topological space (X, τ) is called $fagr\omega$ -open if A^c is $fagr\omega$ -closed.

Example 3.3 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $F_f = \{\phi, X, \{b, c\}, \{a, c\}, \{c\}, \{b\}, \{a\}\}$. It may be easily

checked that the only $fagr\omega$ -closed sets of fine-topological space (X, τ, τ_f) are $\phi, X, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b\}$.

Definition 3.4 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be $fagr\omega$ -continuous if $f^{-1}(V)$ is a $fagr\omega$ -closed set of (X, τ, τ_f) for every fine-closed set V of (Y, τ', τ'_f) .

Example 3.4 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}$, $F_f = \{\phi, X, \{b, c\}, \{c\}, \{b\}, \{a, c\}, \{a\}\}$ and let $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\phi, X, \{1, 2\}\}$, $\tau'_f = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $F'_f = \{\phi, X, \{2, 3\}, \{1, 3\}, \{3\}, \{2\}, \{1\}\}$. We define a map $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It can be easily check that, the function f is $fagr\omega$ -continuous.

Definition 3.5 A function $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ is said to be $fagr\omega$ -irresolute if $f^{-1}(V)$ is an $fagr\omega$ -closed set of (X, τ, τ_f) for every $fagr\omega$ -closed set V of (Y, τ') .

Example 3.5 Let $X = \{a, b, c\}$ be a topological space with the topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}$, $F_f = \{\phi, X, \{b, c\}, \{c\}, \{b\}, \{a, c\}, \{a\}\}$ and let $Y = \{1, 2, 3\}$ with the topology $\tau' = \{\phi, X, \{1, 2\}\}$, $\tau'_f = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $F'_f = \{\phi, X, \{2, 3\}, \{1, 3\}, \{3\}, \{2\}, \{1\}\}$. We define a map $f: (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau'_f)$ by $f(a) = 1, f(b) = 2, f(c) = 3$. It can be easily check that, the function f is $fagr\omega$ -irresolute.

Definition 3.6 Let A be a subset of X . Then $fagr\omega$ -closure of A is the intersection of all $fagr\omega$ -closed sets containing A .

Definition 3.7 Let (X, τ, τ_f) be a fine-topological space. Two non-empty subsets A and B are said to be $fagr\omega$ -separated iff $A \cap fagr\omega-cl(B) = \phi$ and $fagr\omega-cl(A) \cap B = \phi$ i.e., $[A \cap fagr\omega-cl(B)] \cup [fagr\omega-cl(A) \cap B] = \phi$

Definition 3.8 Let (X, τ, τ_f) be a fine-topological space and $A \in X$. A point $x \in X$ is said to be an $fagr\omega$ -adherent point of A if every $fagr\omega$ -open set containing x , contains at least one point of A .

Definition 3.9 If $X = A \cup B$ such that A and B are non-empty $fagr\omega$ -separated sets then A, B form a $fagr\omega$ -separation of X .

Definition 3.10 A fine-topological space (X, τ, τ_f) is said to be $fagr\omega$ -connected if X cannot be written as a union of two disjoint non-empty $fagr\omega$ -open sets. If X is not $fagr\omega$ -connected then it is $fagr\omega$ -disconnected.

Definition 3.11 A topological space (X, τ, τ_f) is said to be $Tfagr\omega$ -space if every $fagr\omega$ -closed set is fine-closed in X .

Theorem 3.1 Two $fagr\omega$ -separated sets are always disjoint.

Proof. Let A and B be $fagr\omega$ -separated sets. Then, we have $A \cap fagr\omega - cl(B) = \phi$ and $fagr\omega - cl(A) \cap B = \phi$

Now, $A \cap B \subseteq fagr\omega - cl(A) \cap B = \phi$. This implies $A \cap B = \phi$. Hence, A and B are disjoint.

Theorem 3.2 Every separated sets are $fagr\omega$ -separated but the converse is not true.

Proof. It is obvious from the definitions.

Theorem 3.3 Two sets are $fagr\omega$ -separated iff they are disjoint and neither of them contains limit point of the other.

Proof. Let A and B be $fagr\omega$ -separated iff $A \cap fagr\omega - cl(B) = \phi$ and $fagr\omega - cl(A) \cap B = \phi$. Now, $A \cap fagr\omega - cl(B) = \phi \Leftrightarrow A \cap (B \cup B_l) = \phi$ where B_l is the limit point of B. $\Leftrightarrow A \cap B = \phi$ and $A \cap B_l = \phi$. A and B are disjoint and A contains no limit point of B. Similarly, $fagr\omega - cl(A) \cap B = \phi$ iff A and B are disjoint and B contains no limit point of A. Hence the proof.

Theorem 3.8 Let A and B be two $fagr\omega$ -separated sets of (X, τ, τ_f) . If $C \subseteq A$ and $D \subseteq B$, then C and D are $fagr\omega$ -separated.

Proof. Let A and B be two $fagr\omega$ -separated sets of a fine-topological space (X, τ, τ_f) . Then, $A \cap fagr\omega - cl(B) = \phi$ and $fagr\omega - cl(A) \cap B = \phi$. Let $C \subseteq A$ and $D \subseteq B$. Then, we have $C \cap fagr\omega - cl(D) = \phi$ and $fagr\omega - cl(C) \cap D = \phi$. Thus C and D are $fagr\omega$ separated.

Theorem 3.9 Two $fagr\omega$ -closed subsets of a fine-topological space (X, τ_f) are $fagr\omega$ -separated iff they are disjoint.

Proof. Since $fagr\omega$ -separated sets are disjoint, $fagr\omega$ -closed separated sets are disjoint. Conversely, let A and B be two disjoint $fagr\omega$ -closed sets. Then $fagr\omega - cl(A) = A$, $fagr\omega - cl(B) = B$ and $A \cap B = \phi$. Consequently, $A \cap fagr\omega - cl(B) = \phi$ and $fagr\omega - cl(A) \cap B = \phi$. Hence, A and B are $fagr\omega$ -separated.

Theorem 3.10 Two $fagr\omega$ -open subsets of a fine-topological space (X, τ_f) are $fagr\omega$ -separated iff they are disjoint.

Proof. Since $fagr\omega$ -separated sets are disjoint, $fagr\omega$ -open separated sets are disjoint. Conversely, let A and B be two disjoint $fagr\omega$ -open sets. Then, we have to prove that $A \cap fagr\omega - cl(B) = \phi$ and $fagr\omega - cl(A) \cap B = \phi$. Suppose $A \cap fagr\omega - cl(B) = \phi$. Let $x \in A \cap fagr\omega - cl(B)$. Then, $x \in A$ and x is an $fagr\omega$ -adherent point of B. Since A is an $fagr\omega$ -open set containing x and x is an $fagr\omega$ -adherent point of B, A contain at least one point of B. Thus $A \cap B = \phi$. This contradicts the fact that A and B are disjoint. Therefore, $A \cap fagr\omega - cl(B) = \phi$. Similarly,

$fagr\omega - cl(A) \cap B = \phi$. Hence A and B are $fagr\omega$ -separated.

4. $fagr\omega$ -Connected Spaces

In this section we define $fagr\omega$ -connectedness in fine topological space.

Definition 4.1 If $X = A \cup B$ such that A and B are non-empty $fagr\omega$ -separated sets then A, B form a $fagr\omega$ -separation of X.

Definition 4.2 A fine-topological space (X, τ_f) is said to be $fagr\omega$ -connected if X cannot be written as a union of two disjoint non-empty $fagr\omega$ -open sets. If X is not $fagr\omega$ -connected then it is $fagr\omega$ -disconnected.

Theorem 4.1 Every $fagr\omega$ -connected space is fine-connected but not conversely.

Proof. Let (X, τ, τ_f) be an $fagr\omega$ -connected space. Suppose that X is not connected. Then $X = A \cup B$, where A and B are disjoint non-empty fine-open sets in (X, τ, τ_f) . Since every fine-open set is $fagr\omega$ -open, A and B are $fagr\omega$ -open. Therefore, $X = A \cup B$. where A and B are disjoint non-empty $fagr\omega$ -open sets in (X, τ, τ_f) : This contradicts the fact that X is $fagr\omega$ -connected and so X is fine-connected

Theorem 4.2 If (X, τ, τ_f) is a $Tfagr\omega$ -space and fine-connected, then X is $fagr\omega$ -connected.

Proof. Obvious from the definition.

Theorem 4.3 A fine-topological space (X, τ, τ_f) is $fagr\omega$ -disconnected if there exists a non-empty proper subset of X which is both $fagr\omega$ -open and $fagr\omega$ -closed.

Proof. Let A be a non-empty proper subset of X which is both $fagr\omega$ -open and $fagr\omega$ -closed. Then clearly A^c is a non-empty proper subset of X which is both $fagr\omega$ -open and $fagr\omega$ -closed. Thus $A \cap A^c = \phi$ and also $X = A \cup A^c$. Thus X is the union of two non-empty $fagr\omega$ -open sets. Hence, X is $fagr\omega$ -disconnected.

Theorem 4.4 A fine-topological space (X, τ, τ_f) is $fagr\omega$ -disconnected iff X is the union of two non-empty disjoint $fagr\omega$ -open sets.

Proof. Let (X, τ, τ_f) be $fagr\omega$ -disconnected. Then, there exists a non-empty proper subset A of X that is both $fagr\omega$ -open and $fagr\omega$ -closed. And therefore, A^c is a non-empty subset of X that is both $fagr\omega$ -open and $fagr\omega$ -closed. Thus $X = A \cup A^c$ and $A \cap A^c = \phi$. This shows that X is the union of two non-empty disjoint $fagr\omega$ -open sets.

Conversely, let X be the union of two non-empty disjoint $fagr\omega$ -open sets A and B. Then $B^c = A$. Now, B is $fagr\omega$ -open, A is $fagr\omega$ -closed. Since B is non-empty, A is a non-empty proper subset of X that is both $fagr\omega$ -open and $fagr\omega$ -closed. Thus X is $fagr\omega$ -disconnected.

Theorem 4.5 For a fine-topological space (X, τ, τ_f) ; the following are equivalent:

- (i) (X, τ, τ_f) is *fragw*-connected.
- (ii) The only subsets of (X, τ, τ_f) which are both *fragw*-open and *fragw*-closed are ϕ and X .

Proof. (i) \Rightarrow (ii) : Let U be an *fragw*-open and *fragw*-closed subset of X . Then U^c is both *fragw*-open and *fragw*-closed in X . Since X is disjoint union of *fragw*-open sets U and U^c , by assumption one of these must be empty. i.e., $U = \phi$ or $U = X$.

(ii) \Rightarrow (i) : Suppose that X is *fragw*-disconnected. Then by Theorem 4.3, there exists a proper subset of X , which is both *fragw*-open and *fragw*-closed, which is a contradiction. Thus X is *fragw*-connected.

Theorem 4.6 If A and B are *fragw*-separated sets of (X, τ, τ_f) and if (Y, τ', τ_f') is an *fragw*-connected subspace of (X, τ, τ_f) , then Y lies entirely within A or B .

Proof. Since A and B are both *fragw*-open in X , the sets $A \cap Y$ and $B \cap Y$ are *fragw*-open in Y . These, two sets are disjoint and their union is Y . If they were both non-empty, they would be *fragw*-separated sets of Y . Therefore, one of them is empty. Hence, Y must lie entirely in A or in B .

Theorem 4.7 If $f : (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau_f')$ is *fragw*-continuous surjection and (X, τ, τ_f) is *fragw*-connected then (Y, τ', τ_f') is fine-connected.

Proof. Suppose that Y is not fine-connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open sets in Y . Since f is *fragw*-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty *fragw*-open sets in X . This contradicts the fact that X is *fragw*-connected. Hence, Y is fine-connected.

Theorem 4.8 If $f : (X, \tau, \tau_f) \rightarrow (Y, \tau', \tau_f')$ is *fragw*-irresolute surjection and (X, τ, τ_f) is *fragw*-connected, then Y is *fragw*-connected.

Proof. Suppose that Y is not *fragw*-connected. Let $Y = A \cup B$ where A and B are disjoint non-empty *fragw*-open set in Y . Since f is *fragw*-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty *fragw*-open sets in X . This contradicts the fact that X is *fragw*-connected. Hence, Y is *fragw*-connected.

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