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On Three Dimensional (ϵ) - Lorentzian Para – Sasakian Manifolds

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Abstract: The object of the present paper is to study three – dimensional (ϵ) - Lorentzian Para – Sasakian Manifolds. We discussed about an (ϵ)- Lorentzian Para – Sasakian 3-Manifold to be an indefinite space form and get necessary and sufficient condition. Also shown that Ricci semi-symmetric (ϵ)- Lorentzian Para – Sasakian 3-Manifold is an indefinite space form. we investigate the necessary and sufficient condition for an (ϵ)- Lorentzian Para – Sasakian 3-Manifold to be locally φ-symmetric.We have also proved that in a three- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds with η- parallel Ricci tensor the scalar curvature τ is constant, Finally we give an example of (ϵ)- Lorentzian Para – Sasakian manifolds which verify all results.

AMS Subject Classification: 53C40, 53C50, 53C55

Keywords: (ϵ) -Lorentzian Para – Sasakian 3 Manifolds, Ricci- semi-symmetric space, locally φ-symmetric space, η- parallel Ricci tensor.

1. Introduction

In [3, 11] K.L Duggle and Bejancu introduced (ϵ) sasakian manifolds. Also Xufeng and Xiaoli [12] proved that every (ϵ)sasakian manifold must be real hypersurface of some indefinite Kaihler manifold. Also [13] R Kumar, R Rani and R Nagaich study (ϵ)sasakian manifolds.Since sasakian manifolds with indefinite metric play significant role in physics.in 1989, K.Matsumoto introduced the notion of Lorentzian Para Sasakian manifold. Recently [2] Rajendra Prasad and Vibha Srivastava defined and study basic properties of (ϵ) Lorentzian Para – Sasakian Manifold with indefinite metric which also include uaual LP sasakian manifold. The present paper is organized as follows:

After introduction in section 2, we introduce the notion of (ϵ) Lorentzian Para – Sasakian Manifold. In section 3, we study the three dimensional (ϵ) Lorentzian Para – Sasakian Manifold and get necessary and sufficient condition for the manifold to be indefinite space form. In section 4 we have shown that three dimensional Ricci semi symmetric (ϵ) Lorentzian Para – Sasakian Manifold is a manifold of indefinite space form. In section 5, a necessary and sufficient condition for an (ϵ) Lorentzian Para – Sasakian 3 Manifolds to be locally φ- symmetric is obtained. Section 6, deals with some result on (ϵ) Lorentzian Para – Sasakian 3 Manifolds with η-parallel Ricci tensor. In last section we give an example which verifies all results.

2. (ϵ) -Lorentzian Para – Sasakian Manifolds

An n- dimensional differentiable manifold is called (ϵ) - Lorentzian Para – Sasakian Manifold if the following condition hold [2]

(2.1) $\varphi^2 = I + \eta(X)\xi, \eta(\xi) = -1$ (2.2) $g(\xi,\xi) = \epsilon, g(X,\xi) = \epsilon\,\eta(X)$
(2.3) $g(\varphi X, \varphi Y) = g(X,Y) + \epsilon\,\eta(X)\,\eta(Y)$

Where ϵ is 1 or -1 according as is space –like or time like vector field. Also in (ϵ)- Lorentzian Para – Sasakian Manifold, we have [2]

(2.4) $(\nabla_X\varphi)Y = g(X,Y)\xi + \epsilon\,\eta(Y)X + 2\epsilon\,\epsilon\,\eta(X)\,\eta(Y)$
(2.5) $\nabla_X\xi = \epsilon\,\varphi X$
(2.6) $\emptyset(X,Y) = g(\varphi X, Y) = g(X, \varphi Y) = (\nabla_X\eta)Y$

From (2.6) $\emptyset(\xi, Y) = 0$

Where ∇ denotes the operator of covariant of differentiation with respect to lorentzian metric g.

In an (ϵ)-Lorentzian Para – Sasakian Manifold, the Riemannian curvature R and the Ricci tensor S satisfy the following equations [2]:

(2.7) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$
(2.8) $\eta(R(X,Y)Z) = \epsilon\,(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))$
(2.9) $R(\xi,X)Y = \epsilon\,g(X,Y)\xi - \eta(Y)X$
(2.10) $g(R(X,Y)Z,\xi) = \epsilon\,g(Y,Z)g(X,\xi) - \epsilon\,g(X,Z)g(Y,\xi)$
(2.11) $S(X,\xi) = (n-1)\eta(X)$

Definition 2.1: A (ϵ) Lorentzian Para – Sasakian Manifold is said to be Ricci- Semi-symmetric if the Ricci Tensor S Satisfies

(2.12) $R(X,Y).S = 0$

Where $R(X,Y).S$ denotes the derivation of the tensor algebra at each point of the manifold.

Definition 2.2: A (ϵ) Lorentzian Para – Sasakian Manifold is said to be locally φ- symmetric if

(2.13) $\varphi^2(\nabla_W R)(X,Y)Z = 0,$

For all vector fields $W, X\,Y, Z$ orthogonal to ξ.

This notion was introduced by Takahashi for Sasakian manifold [1].

Definition 2.3: A (ϵ) Lorentzian Para – Sasakian Manifold is said to be η- parallel if it is satisfies

(2.14) $(\nabla_Z S)(\varphi X, \varphi Y) = 0$, for all vector fields X, Y, Z.

The notion of η-parallelity for Sasakian manifolds was introduced by Kon [8]

3. Three Dimensional (ϵ) Lorentzian Para – Sasakian Manifolds

It is known that in a semi – Riemannian 3-manifold

(3.1) $$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{\tau}{2}[g(Y,Z)X - g(X,Z)Y]$$

Where Q is the Ricci operator and τ is scalar curvature of the manifold.

Putting $Z = \xi$ in (3.1) and using (2.2), (2.7) and (2.12), we have

(3.2) $\eta(Y)QX - \eta(X)QY = \frac{1}{2}(\tau - 2\epsilon)(\eta(Y)X - \eta(X)Y)$

Putting $Y = \xi$ in (3.2) and using (2.1) and (2.11) we get

(3.3) $QX = \frac{1}{2}(\tau - 2\epsilon)X - \frac{1}{2}(\tau - 6\epsilon)\eta(X)\xi$

Therefore

(3.4) $S(X,Y) = \frac{1}{2}(\tau - 2\epsilon)\, g(X,Y) - \frac{1}{2}\epsilon(\tau - 6\epsilon\eta X\eta Y$

Using (3.3) and (3.4) in (3.1), we obtain

(3.5) $R(X,Y)Z = \left(\frac{\tau}{2} - 2\epsilon\right)\{g(Y,Z)X - g(X,Z)Y\} + \left(\frac{\tau}{2} - 3\epsilon\right)\{\, g(Y,Z)\,\eta(X)\xi - g(X,Z)\,\eta(Y)\xi + \epsilon\,\eta(Y)\,\eta(Z)\,X - \epsilon\,\eta(X)\,\eta(Z)\,Y\}$

If an (ϵ)- Lorentzian Para – Sasakian manifold is a space of constant curvature then it is an indefinite space form. Let us consider (ϵ)- Lorentzian Para – Sasakian 3-Manifold be an indefinite space form then

(3.6) $R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, X, Y, Z, \in \chi(M)$

Where c is the constant curvature of the manifold. In view of definition of Ricci curvature and from (3.6), we have

(3.7) $S(X,Y) = 2cg(X,Y)$

And in view of definition of scalar curvatur, we have from (3.7)

(3.8) $\tau = 6c$

From (3.7) and (3.8), we have

(3.9) $S(X,Y) = \frac{\tau}{3}\, g(X,Y)$

By putting $X = Y = \xi$ in (3.4) and using (3.9) we obtain

$\tau = 6\epsilon$

Conversely, if $\tau = 6\epsilon$ them from (3.5) we can easily seen that manifold is an indefinite space form.

Therefore we have following Lemma:

Lemma 3.1: An (ϵ)- Lorentzian Para – Sasakian 3-Manifold be an indefinite space form if and only if the scalar curvature $\tau = 6\epsilon$.

4. Three Dimensional Ricci- Semisymmetric (ϵ)- Lorentzian Para – Sasakian Manifolds

Let M be the Ricci semi symmetric (ϵ)-Lorentzian Para – Sasakian 3-Manifold. Then in view of definition (2.1), we can write

(4.1) $(R(X,Y),S)(U,V) = R(X,Y)S(U,V) - S(R(X,Y)U,V) - S(U,R(X,Y)V)$

Then from (2.12), we have

(4.2) $S(R(X,Y)U,V) - S(U,R(X,Y)V) = 0$

Putting $X = \xi$ in (4.2) and using (2.9) and (2.11) we get

(4.3) $2\epsilon g(Y,U)\eta(V) - S(Y,V)\eta(U) + 2\epsilon g(Y,V)\eta(U) - S(Y,U)\eta(V) = 0$

Let $\{e_1, e_2, \xi\}$ be an orthogonal basis of the tangent space at each point of 3- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds then by putting $Y = U = e_i$ in (4.2), we obtain

(4.4) $\eta(V)[2\epsilon g(e_i,e_i) - S(e_i,e_i)] = 0$

Since $S(e_i,e_i) = \left[\frac{\tau}{2} - \epsilon\right] g(e_i,e_i)$, therefore from (4.4), we get

$$\left[3\epsilon - \frac{\tau}{2}\right] g(e_i,e_i) = 0$$

Which implies

(4.5) $\tau = 6\epsilon$, since $g(e_i,e_i) \neq 0$

Therefore in view of Lemma (3.1), the manifold is an indefinite space form.

Then we state the following

Theorem 4: A Ricci semi symmetric (ϵ)- Lorentzian Para – Sasakian 3-Manifold is an indefinite space form.

5. Locally φ-Symmetric Three Dimensional (ϵ) Lorentzian Para – Sasakian Manifolds

Analogous to the definition (2.2), let us consider a three dimensional (ϵ)- Lorentzian Para – Sasakian manifold. Differentiating (3.5) covariantly with respect to W, we get

$$(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{d\tau(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \epsilon\eta(Y)\eta(Z)X - \epsilon\eta(X)\eta(Z)Y] + \left[\frac{\tau}{2} - 3\epsilon\right][g(Y,Z)(\nabla_W\eta)(X)\xi - g(X,Z)(\nabla_W\eta)(Y)\xi + \epsilon g(Y,Z)\eta(X)\varphi W - \epsilon g(X,Z)\eta(Y)\varphi W] + \epsilon(\nabla_W\eta)(Y)\eta(Z)X - \epsilon(\nabla_W\eta)(X)\eta(Z)Y + \epsilon\eta(Y)(\nabla_W\eta)(Z)X - \epsilon\eta(X)(\nabla_W\eta)(Z)Y]$$

On account of X,Y,Z,W to orthogonal to ξ, then above equation becomes

$$(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \left[\frac{\tau}{2} - 3\epsilon\right][g(Y,Z)(D_W\eta)(X)\xi - g(X,Z)(D_W\eta)(Y)\xi]$$

Using (2.6) we get

$$(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y] + \left[\frac{\tau}{2} - 3\epsilon\right][g(Y,Z)\emptyset(W,X)\xi + g(X,Z)\emptyset(W,Y)\xi]$$

From above it follows that

$$\varphi^2(\nabla_W R)(X,Y)Z = \frac{d\tau(W)}{2}[g(Y,Z)X - g(X,Z)Y]$$

Hence we state the following theorem.

Theorem 5.1: A three dimensional (ϵ)- Lorentzian Para – Sasakian manifold is locally φ-symmetric if an only if the scalar curvature τ is constant.

Again if the manifold is Ricci semi-symmetric then we have seen that $\tau = 6\epsilon$ that is τ is constant, then we have

Theorem 5.2: A three dimensional Ricci-semi-symmetric (ϵ)- Lorentzian Para – Sasakian manifold is locally φ-symmetric

6. Three Dimensional (ϵ) Lorentzian Para – Sasakian Manifolds with η-Parellel Ricci Tensor

Now let us consider three dimensional (ϵ)- Lorentzian Para – Sasakian manifold with η – parallel Ricci tensor. Then from (3.4), we have

(6.1) $S(\varphi X,\varphi Y) = \left[\frac{\tau}{2} - \epsilon\right] g(\varphi X,\varphi Y)$

Then using (2.3), we get

(6.2) $S(\varphi X,\varphi Y) = \left[\frac{\tau}{2} - \epsilon\right]\{g(X,Y) + \epsilon\,\eta(X)\eta(Y)\}$

Differentiating (6.2), covariantly along Z and using (2.6), we have

(6.3) $(\nabla_Z S)(\varphi X,\varphi Y) = \frac{d\tau(Z)}{2}(g(X,Y) + \epsilon\eta(X)\eta(Y)) + \left[\frac{\tau}{2} - \epsilon\right](\epsilon\eta(Y)\emptyset(Z,X) + \epsilon\eta(X)\emptyset(Z,Y))$

Since M is three- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds with η- parallel Ricci tensor, then in view of definition (2.14), we have

(6.4) $0 = \frac{d\tau(Z)}{2}(g(X,Y) + \epsilon\eta(X)\eta(Y)) + \left[\frac{\tau}{2} - \epsilon(\epsilon\eta Y\emptyset Z,X + \epsilon\eta X\emptyset Z,Y)\right.$

Let $\{e_1, e_2, \xi\}$ be an orthogonal basis of the tangent space $T_pM, p \in M$ at each point of 3- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds then by putting $X = Y = e_i, 1 \le i \le 3$, in (6.3), we obtain

$$d\tau(Z) = 0, Z \in \chi(M)$$

Thus we have following proposition

Proposition 6.1: If a three- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds has η- parallel Ricci tensor then scalar curvature τ is constant.

Therefore in view of (5.1) and proposition (6.1) we have following

Theorem 6.2: An three- dimensional (ϵ)- Lorentzian Para – Sasakian manifolds with η- parallel Ricci tensor is locally φ-symmetric.

7. Example

We consider the three dimensional manifold $M = [(x,y,z)] \in R^3, z \neq 0$ where (x,y,z) are standard coordinates in R^3. The vector fields

$$e_1 = e^z\frac{\partial}{\partial y}, e_2 = e^z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), e_3 = \frac{\partial}{\partial z}$$

Are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0$$
$$g(e_1,e_1) = g(e_2,e_2) = \epsilon, g(e_3,e_3) = -\epsilon$$

That is, form of the metric becomes

$g = \frac{1}{(e^z)^2}(dy)^2 - (dz)^2$, which is a Lorentzina metric.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$for any $Z \in \chi(M)$.let φ be the (1, 1) tensor field defined by $\varphi e_1 = -e_1, \varphi e_2 = -e_2, \varphi e_3 = 0$. then using the linearity property of φ and g, we have

$$\eta(e_3) = -1, \varphi^2(Z) = Z + \eta(Z)e_3$$

and $g(\varphi Z, \varphi W) = g(Z, W) + \epsilon\eta(Z)\eta(W)$ for any Z,W $\in \chi(M)$, therefore $e_3 = \xi$, the structure (φ, ξ, η, g) defines a (ϵ) Lorentzian Para contact structure on M. Let ∇ be the Levi Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. then we have $[e_1, e_2] = 0, [e_1, e_3] = -\epsilon e_1, [e_2, e_3] = -\epsilon e_2$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \quad (7.1)$$

Using (7.1) for lorentzian metric g, we can easily calculate that

$$\nabla_{e_1} e_1 = -\epsilon e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = -\epsilon e_1,$$
$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -\epsilon e_3, \nabla_{e_2} e_3 = -\epsilon e_2,$$
$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0,$$

From above we also have $\nabla_X \xi = \epsilon\varphi X$. Hence the structure (φ, ξ, η, g) defines a (ϵ)- Lorentzian Para-Sasakian manifold. Now using the above result, we have

$$R(e_1, e_2)e_3 = 0, R(e_2, e_2)e_3 = -e_2,$$
$$R(e_1, e_3)e_3 = -e_1, R(e_1, e_2)e_2 = e_1,$$
$$R(e_2, e_3)e_2 = -e_3, R(e_1, e_2)e_1 = e_2, \quad (7.2)$$
$$R(e_3, e_1)e_1 = e_3, R(e_2, e_1)e_1 = e_2,$$
$$R(e_3, e_2)e_2 = e_3$$

From which it follows that

$$\varphi^2(\nabla_W R)(X, Y,)Z = 0,$$

Hence, the three Dimensional $(\epsilon)-$ Lorentzian Para Sasakian Manifolds is locally φ-Symmetric.

Also from the above expression of the curvature tensor we obtain

$$S(e_1, e_1) = S(e_2, e_2) = 0, S(e_3, e_3) = -2 \quad (7.3)$$

Hence $\tau = -2$,

Which is constant. therefore the theorem (5.1) is verified. Also from the expression from the Ricci tensor, we find manifold is Ricci Semi symmetric. From (7.2) we can easily see that manifold is manifold of constant curvature 1. Hence theorem (5.2) is verified.

Finally, from (7.3), we get

$(\nabla_Z S)(\varphi X, \varphi Y) = 0$, for all X, Y, Z. Therefore theorem (6.2) is verified.

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