Fixed Point Theorem for Compatible Mapping in Fuzzy Metric Space

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Abstract: The present paper we establish a common fixed point theorem for compatible pair of self maps in a fuzzy metric space.

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1. Introduction

The concept of fuzzy sets was initiated by Zadeh [27] in 1965 and the concept of fuzzy metric space was introduced by Kramosil and Michalek [12]. Grabiec [7] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [25] for a pair of commuting mappings. Also, George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Also, Jungck and Rhoades [10] defined a contraction principle in the setting of the fuzzy metric space by Kramosil and Michalek [12]. Grabiec [7] proved the following conditions: for all \( x, y, z \in X \), and \( s, t > 0 \),

(i) \( M(x, y, 0) = 0 \), \( M(y, y, t) > 0 \),

(ii) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),

(iii) \( M(x, y, t) = M(y, x, t) \),

(iv) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \),

(v) \( M(x, y, \cdot : [0, \infty) \to [0, 1] \) is left continuous.

In this case, \( M \) is called a fuzzy metric on \( X \) and the function \( M(x, y, t) \) denotes the degree of nearness between \( x \) and \( y \) with respect to \( t \).

Also, we consider the following condition in the fuzzy metric space \((X, M, \ast)\):

\( \lim_{t \to 0} M(x, y, t) = 1 \), for all \( x, y \in X \).

It is important to note that every metric space \((X, d)\) induces a metric space \((X, M, \ast)\) where \( \ast = \min \{a, b\} \) and for all \( a, b \in X \), we have \( M(x, y, t) = \frac{t}{1 + d(x, y)} \), for all \( t > 0 \), and \( M(x, y, 0) = 0 \), so-called the fuzzy metric space induced by the metric \( d \).

DEFINITION 1.3[6]) The triplet \((X, M, \ast)\) is called a fuzzy metric space (shortly, a FM-space) if, \( X \) is an arbitrary set, \( \ast \) is a continuous t-norm and \( M \) is a fuzzy set on \( X \times [0, 1] \) satisfying

(i) \( M(x, y, 0) = 0 \), \( M(y, y, t) > 0 \),

(ii) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),

(iii) \( M(x, y, t) = M(y, x, t) \),

(iv) \( M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \),

(v) \( M(x, y, \cdot : [0, \infty) \to [0, 1] \) is left continuous.

The purpose of this paper is to obtain a common fixed point theorem for compatible pair of self mappings in fuzzy metric space.

We have used the following notions:

DEFINITION 1.1[27]) Let \( X \) be any set. A fuzzy set \( A \) in \( X \) is a function with domain \( X \) and values in \([0, 1]\).

DEFINITION 1.2[6]) A binary operation \( \ast : [0, 1] \times [0, 1] \to [0, 1] \) is called a continuous t-norm if, \((0, 0), \ast)\) is an abelian topological monoid with unit 1 such that a \( \ast b \leq c \) whenever \( a \leq c \) and \( b \leq d \), for all \( a, b, c, d \in [0, 1] \).

For an example: \( a \ast b = ab, a \ast b = \min \{a, b\} \).
sequence such that \( \lim n \to \infty A_n = \lim n \to \infty B_n = p \), for some \( p \) in \( X \).

**LEMMA 1.11** ([20]) Let \((X, M, *)\) be a fuzzy metric space. If there exists \( k \in (0, 1) \) such that \( M(x, y, k t) \geq M(x, y, t) \) for all \( x, y \) in \( X \), then \( x = y \).

**PROPOSITION 1.12:** Let \( A \) and \( B \) be compatible, self mappings of a fuzzy metric space \( X \).

(1) If \( A y = B y \) then \( AB y = BA y \).

(2) If \( A_n, B \to y \), for some \( y \) in \( X \) then \( A B x_n \to A y \) if \( A \) is continuous.

(3) If \( A \) and \( B \) are continuous at \( y \) then \( A y = B y \) and \( AB x_n \to B y \).

PROOF: (1) Let \( A y = B y \) and \( \{x_n\} \) be a sequence in \( X \) such that \( x_n \to y \) for all \( n \). Then \( A x_n \to A y \) and \( B x_n \to B y \). Now by the compatibility of \( A \) and \( B \), we have \( M(A B x_n, B A y, t) = M(A B x_n, B A y, t) = 1 \) which yields \( A B x_n \to A y \).

(2) If \( A_n \to y \), for some \( y \) in \( X \) then \( A B x_n \to A y \) if \( A \) is continuous.

(3) If \( A \) and \( B \) are continuous then from (a) we have \( B A x_n \to A y \). By the uniqueness of the limit \( A y = B y \). Hence \( AB x_n \to B y \).

2. Main Results

**THEOREM 2.1.** Let \((X, M, *)\) be a complete fuzzy metric space with additional condition (vi) and with \( a \geq a \geq 1 \) for all \( a \in [0, 1] \). Let \( A, B, S \) and \( T \) be mappings from \( X \) into itself such that

(i) \( A(X) \subseteq T(X), B(X) \subseteq S(X) \)

(ii) One of \( A, B, S \) or \( T \) is continuous,

(iii) \((A, S) \) and \((B, T)\) are compatible pairs of mappings,

(iv) \( M(A x, B y, t) \geq \phi(\min\{M(S y, T y, t), M(A x, T y, t), M(S y, B y, (2 - \alpha t), t)\}) \) for all \( x, y \in X, \alpha \in (0, 2) \) and \( t > 0 \),

where \( \phi : [0, 1] \to [0, 1] \) is a continuous function such that \( \phi(t) > t \) for each \( 0 < t < 1 \).

Then, setting \( x = u \) and \( y = x 2n+1 \) in contractive condition (iv) with \( \alpha = 1 \), we get

\[
M(A u, B x 2n+1, t) \geq \phi(\min\{M(S x 2n+1, T x 2n+1, t), M(A x 2n+1, T x 2n+1, t), M(S x 2n+1, B x 2n+1, t)\})
\]

\[
= 1. \text{ If } \phi(\min\{M(S x 2n+1, T x 2n+1, t), M(A x 2n+1, T x 2n+1, t), M(S x 2n+1, B x 2n+1, t)\}) = 1 \text{ whenever } x = u \text{ and } y = x 2n+1 \text{ in contractive condition (iv) with } \alpha = 1, \text{ we get}
\]

\[
M(A u, B x 2n+1, t) \geq \phi(\min\{M(S x 2n+1, T x 2n+1, t), M(A x 2n+1, T x 2n+1, t), M(S x 2n+1, B x 2n+1, t)\}) = 1 \text{ whenever } x = u \text{ and } y = x 2n+1 \text{ in contractive condition (iv) with } \alpha = 1, \text{ we get}
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M(A u, B x 2n+1, t) \geq \phi(\min\{M(S x 2n+1, T x 2n+1, t), M(A x 2n+1, T x 2n+1, t), M(S x 2n+1, B x 2n+1, t)\}) = 1 \text{ whenever } x = u \text{ and } y = x 2n+1 \text{ in contractive condition (iv) with } \alpha = 1, \text{ we get}
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M(A u, B x 2n+1, t) \geq \phi(\min\{M(S x 2n+1, T x 2n+1, t), M(A x 2n+1, T x 2n+1, t), M(S x 2n+1, B x 2n+1, t)\}) = 1 \text{ whenever } x = u \text{ and } y = x 2n+1 \text{ in contractive condition (iv) with } \alpha = 1, \text{ we get}
\]
condition (iv) with \( \alpha = 1 \), one can show that \( Au = u \). Therefore, we have \( u = Au = Bu = Tu = Su \). Hence, the point \( u \) is a common fixed point of A, B, S and T.

3. Uniqueness

The uniqueness of the common fixed point of the mappings A, B, S and T be easily verified by using (iv). In fact, if \( u_0 \) be another fixed point for mappings A, B, S and T. Then, for \( \alpha = 1 \), we have

\[
M(u, u_0, t) = M(Au, Bu_0, t) \geq \phi(\min\{M(Su, Tu_0, t), M(Au, Tu_0, t), M(Su, Bu_0, t)\}) \\
\geq \phi(M(u, u_0, t)) > M(u, u_0, t),
\]

and hence, we get \( u = u_0 \). This completes the proof of the theorem.

References


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