Fixed Point Theorem for Compatible Mapping in Fuzzy Metric Space

Raghu Nandan Patel¹, Manoj Kumar Tiwari²

¹Government Naveen College, Balrampur, Chhattisgarh, India
²Government Girls Polytechnic College, Bilaspur, Chhattisgarh, India

Abstract: The present paper we establish a common fixed point theorem for compatible pair of self maps in a fuzzy metric space.

2000 Mathematics Subject Classification: 54H25, 47H10.

Key Words and phrases: Common fixed point, Fuzzy metric space, compatible maps.

1. Introduction

The concept of fuzzy sets was initiated by Zadeh [27] in 1965 and the concept of fuzzy metric space was introduced by Kramosil and Michalek [12]. Grabiec [7] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [25] for a pair of commuting mappings. Also, George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. Also, Jungck and Rhoades [10] defined a generalizing the concept of probabilistic metric space to fuzzy metric space by replacing continuity condition with reciprocally continuity maps. Balasubramaniam et. al. [9] has proved a common fixed point theorem for compatible pair of self mappings in fuzzy metric space. Jha et.al.[1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space. Jha et.al.[9] has proved a common fixed point theorem for four self mappings in fuzzy metric space under the weak contractive conditions. Also, B. Singh and S. Jain [23] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type (α), compatible map of type (β) and obtained some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [7]. As a generalization of fixed point results of Singh and Jain [23], Mishra et. al.[15] proved a fixed point theorems in complete fuzzy metric space by replacing continuity condition with reciprocally continuity maps.

The purpose of this paper is to obtain a common fixed point theorem for compatible pair of self mappings in fuzzy metric space.

We have used the following notions:

DEFINITION 1.1[27]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

DEFINITION 1.2[6]) A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if, (i) [0, 1], * is an abelian topological monoid with unit 1 such that a * b ≤ c whenever a ≤ c and b ≤ d, for all a, b, c, d in [0, 1].

For an example: a * b = ab, a * b = min {a, b}.

DEFINITION 1.3[6]) The triplet (X, M, *) is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X×X × [0, 1] satisfying the following conditions: for all x, y, z in X and s, t > 0,

(i) M(x, y, 0) = 0, M(x, y, t) > 0;
(ii) M(x, y, t) = 1 for all t > 0 if and only if x = y;
(iii) M(x, y, t) = M(y, x, t);
(iv) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),
(v) M(x, y, ·) : [0, ∞) → [0, 1] is left continuous.

In this case, M is called a fuzzy metric on X and the function M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Also, we consider the following condition in the fuzzy metric space (X, M, *):

(vi) \lim_{t \to 0} M(x, y, t) = 1, for all x, y ∈ X.

It is important to note that every metric space (X, d) induces a fuzzy metric space (X, M, *) where a * b = min {a, b} and for all a, b ∈ X, we have M(x, y, t) = \frac{1}{1 + \frac{d(x, y)}{t}} for all t > 0, and M(x, y, 0) = 0, so-called the fuzzy metric space induced by the metric d.

DEFINITION 1.4[6]) A sequence {xn} in a fuzzy metric space (X, M, *) is called a Cauchy sequence if, \lim_{n \to \infty} M(xn+p, xn, t) = 1 for every t > 0 and for each p > 0.

A fuzzy metric space (X, M, *) is complete if, every Cauchy sequence in X converges in X.

DEFINITION 1.5[6]) A sequence {xn} in a fuzzy metric space (X, M, *) is said to be convergent to x in X if, \lim_{n \to \infty} M(xn, x, t) = 1, for each t > 0.

It is noted that since * is continuous, it follows from the condition (iv) of Definition (1.3) that the limit of a sequence in a fuzzy metric space is unique.

DEFINITION 1.6[1]) Two self mappings A and B of a fuzzy metric space (X, M, *) are said to be compatible if, \lim_{n \to \infty} M(ABxn, BAxn, t) = 1 whenever {xn} is a
sequence such that \( \lim_{n \to \infty} A x_n = \lim_{n \to \infty} B x_n = p \), for some \( p \) in \( X \).

**Lemma 1.11** Let \((X, M, \ast)\) be a fuzzy metric space. If there exists \( k \in (0, 1) \) such that \( M(x, y, k t) \geq M(x, y, t) \), then \( x = y \).

**Proposition 1.12** Let \( A \) and \( B \) be compatible, self mappings of a fuzzy metric space \( X \).

1. If \( A \) and \( B \) are continuous then from (a) we have \( B A x_n \rightarrow \) \( M(AB) \) sequence \( \{x_n\} \) a common fixed point in \( X \).
2. If \( A \) and \( B \) are continuous then for some \( y \) in \( X \) then \( A B x_n \rightarrow Ay \) if \( A \) is continuous.

**Proof:** (1) Let \( A y = B y \) and \( \{x_n\} \) be a sequence in \( X \) such that \( x_n = y \) for all \( n \). Then \( A x_n \), \( B x_n \rightarrow y \) for some \( y \) in \( X \) then \( (a) \) \( B A x_n \rightarrow Ay \) if \( A \) is continuous.

2. **Main Results**

**Theorem 2.1** Let \((X, M, \ast)\) be a complete fuzzy metric space with additional condition (vi) and with \( a \geq 1 \) for all \( a \in [0, 1] \). Let \( A, B, S \) and \( T \) be mappings from \( X \) into itself such that

(a) \( A(X) \subseteq T(X), B(X) \subseteq S(X) \)

(b) One of the \( A, B, S \) or \( T \) is continuous,

(c) \((A, S)\) and \((B, T)\) are compatible pairs of mappings,

(d) \( M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Ty, t), M(Sx, By, (2 - q)t), \ldots\}) \) for all \( x, y \in X \), \( \alpha > 0 \), \( 0 < t < 1 \). Where \( \phi : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( \phi(t) \rightarrow t \) for some \( 0 < t < 1 \).

Then, \( A \), \( B \), \( S \) and \( T \) have a unique common fixed point in \( X \).

**Proof:** Let \( x_0 \in X \) be an arbitrary point. Then, since \( A(X) \subseteq T(X), B(X) \subseteq S(X) \), there exists \( x_1, x_2 \in X \) such that \( A x_0 = T x_1 \) and \( B x_1 = S x_2 \). Inductively, we construct the sequences \( \{x_n\} \) and \( \{x_n\} \) in \( X \) such that \( y_2n = A x_{2n} = T x_{2n+1} \) and \( y_2n+1 = B x_{2n+1} = S x_{2n+2} \), for \( n = 0, 1, 2, \ldots \).

Now, we put \( a = 1 - q < 1 \) and \( q \) such that \( \phi(t) = \phi(q) \), for all \( 0 < t < 1 \). By (iv), we have \( M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(x_{2n}, x_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, x_{2n+1}, t), M(x_{2n}, y_{2n+1}, t)\}) \)

\( \geq \phi(\min\{M(x_{2n}, x_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, x_{2n+1}, t), M(x_{2n}, y_{2n+1}, t)\}) \)

\( \geq \phi(\min\{M(x_{2n}, x_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, x_{2n+1}, t), M(x_{2n}, y_{2n+1}, t)\}) \)

Since \( t \)-norm \( \ast \) is continuous, letting \( q \rightarrow 1 \), we have
condition (iv) with \( \alpha = 1 \), one can show that \( Au = u \). Therefore, we have \( u = Au = Bu = Tu = Su \). Hence, the point \( u \) is a common fixed point of \( A, B, S \) and \( T \).

3. Uniqueness

The uniqueness of a common fixed point of the mappings \( A, B, S \) and \( T \) be easily verified by using (iv). In fact, if \( u_0 \) be another fixed point for mappings \( A, B, S \) and \( T \). Then, for \( \alpha = 1 \), we have

\[
M(u, u_0, t) = M(Au, Bu_0, t) \geq \phi(\min\{M(Su, Tu_0, t), M(Au, Tu_0, t), M(Su, Bu_0, t)\}),
\]

\[
\geq \phi(M(u, u_0, t)) > M(u, u_0, t),
\]

and hence, we get \( u = u_0 \). This completes the proof of the theorem.

References


Author Profile

Raghu Nandan Patel is in Department of Mathematics, Government Naveen College, Balrampur, Chhattisgarh, India

Manoj Kumar Tiwari is in Department of Mathematics, Government Girls Polytechnic College, Bilaspur, Chhattisgarh, India