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The Zeros of Polar Derivative of Polynomials with Respect to Real Number

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Abstract: ***In this paper we obtain the size of the disc in which the zeros of polar derivatives of polynomial of degree n with real coefficients with respect to a real α lie.***

Keywords: zeros, polar derivatives, polynomials, real α.

1. Introduction

To estimate the zeros of a polynomial is a long standing problem. It is an interesting area of research for many engineers as well as mathematicians and many results on the topic are available in the literature.

If $P(z)=\sum_{i=0}^{n} a_i z^i$, be a polynomial of degree n then Polar Derivative of the polynomial P(z) with respect to α, where α can be real or complex number, is defined as

$$D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z).$$

It is a polynomial of degree up to n-1. The polynomial $D_\alpha P(z)$ generalizes the ordinary derivative, in the sense that $\lim_{\alpha\to\infty} D_\alpha P(z)/\alpha = P'(z)$.

The well-known results on Eneström-Kakeya theorem (see [1,2]) in theory of distribution of zeros of polynomials are the following.

Theorem (1): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that
$0<a_0 \le a_1 \le \ldots \le a_n$.
Then all the zeros of P(z) lie in $|z| \le 1$.
A. Joyal , G. Labelle and Q.I.Rahman [3] obtained the following generalization, by considering the coefficients to be real instead of being only positive.

Theorem (2): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that
$a_0 \le a_1 \le \ldots \le a_n$.
Then all the zeros of P(z) lie in $|z| \le |a_n|^{-1}\{a_n - a_0 + |a_0|\}$.
This paper we prove the following results.
Aziz and Zargar [1] relaxed the hypothesis of Enestr□om-Kakeya theorem in
a different way and proved the following results.

Theorem (3): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that k≥1 ,
$0< a_0 \le a_1 \le \ldots \le a_{n-1} \le k a_n$.
Then all the zeros of P(z) lie in $|z+k-1|\le k$.

Theorem (4): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that k≥1 ,
$a_0 \le a_1 \le \ldots \le a_{n-1} \le k a_n$.
Then all the zeros of P(z) lie in $|z+k-1|\le |a_n|^{-1}\{k a_n - a_0 + |a_0|\}$.

2. Zeros of Polar Derivative of Polynomial P(z)

This section is concerned with the location of the zeros of the polar derivative of a polynomial with real coeffcients with respect to a real number.

Theorem (5): Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that k≥1 ,
$a_0 \le a_1 \le \ldots \le a_{n-1} \le k a_m \ge a_{m+1} \ge \ldots a_{n-1} \ge a_n$
$(n-i)a_i \le a_{i+1}$, i= 0, 1, 2, . . .m-1
$j a_j \le (j-1) a_{j-1}$, j= m+1,. . . n.
Then for a real α the polar derivatives of P(z) with respect to α has at most n -1 zeros and they lie in

$$|z| \le |a_{n-1} + \alpha n a_n|^{-1}\{-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + k\alpha m a_m - n a_0 - \alpha a_1 + |n a_0 + \alpha a_1|\}.$$

Proof: Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n$ be a polynomial of degree n.
Then the polar derivative of P(z) is given by $D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)$.

Then
$$D_\alpha P(z) == [n a_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2]\,z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$$
$$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots\ldots$$
$$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}.$$

Now consider the polynomial $Q(z) = (1-z)\,D_\alpha P(z)$ so that
$$Q(z) = -[a_{n-1} + \alpha n a_n]z^n + [a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}]z^{n-1} + \ldots$$
$$+ [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$$
$$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m$$
$$+ [(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$$
$$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - n a_0 - \alpha a_1]z + [n a_0 + \alpha a_1].$$
Now if $|z|>1$ then $|z|^{-(n-i)} < 1$ for i = 1, 2, 3, …..n-1

Further
$$|Q(z)| \ge |a_{n-1} + \alpha n a_n||z|^n - \{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}|\,|z|^{n-1} + \ldots + |(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\,|z|^{m+1}$$

$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|\,|z|^m$
$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\,|z|^{m-1}$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|$
$+|na_0 + \alpha a_1|\}$.
$\geq |a_{n-1} + \alpha n a_n|\,|z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}|\,|z|^{-1} + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|\,|z|^{-(n-m-2)}$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|\,|z|^{-(n-m-1)}$
$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}|\,|z|^{-(n-m)} + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^{-(n-3)} + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|\,|z|^{-(n-2)} + |na_0 + \alpha a_1|\,|z|^{-(n-1)}\}]$.

$\geq |a_{n-1} + \alpha n a_n|\,|z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$
$+|(n-m+1)a_{m-1} + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1| + |na_0 + \alpha a_1|\}]$.

$\geq |a_{n-1} + \alpha n a_n|\,|z|^{n-1}[|z| - |a_{n-1} + \alpha n a_n|^{-1}\{|a_{n-1} + \alpha n a_n - 2a_{n-2} - \alpha(n-1)a_{n-1}| + |2a_{n-2} + \alpha(n-1)a_{n-1} - 3a_{n-3} - \alpha(n-2)a_{n-2}| + \ldots$
$+|(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - k(n-m)a_m - \alpha(m+1)a_{m+1} + k(n-m)a_m - (n-m)a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$
$+|(n-m+1)a_{m-1} + \alpha k m a_m - \alpha k m a_m + \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}| + \ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1|$
$+|na_0 + \alpha a_1|\}]$.

$\geq |a_{n-1}+\alpha n a_n||z|^{n-1}(|z| - |a_{n-1}+\alpha n a_n|^{-1}\{[2a_{n-2}+\alpha(n-1)a_{n-1} - a_{n-1} - \alpha n a_n] + [3a_{n-3} + \alpha(n-2)a_{n-2} - 2a_{n-2} - \alpha(n-1)a_{n-1}] + \ldots + [k(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m-1)a_{m+1} - \alpha(m+2)a_{m+2}] + (k-1)(n-m)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + [(n-m+1)a_{m-1} + \alpha k m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1} + (1-k)|a_m|] + \ldots + [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2] + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1] + |na_0 + \alpha a_1|\})$

$\geq |a_{n-1}+\alpha n a_n||z|^{n-1}(|z| - |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$
$+(n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|\})$
> 0 if $|z| > |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m|$
$+|(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m|$
$+(n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)$

This shows that if $|z| > |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)$, then $Q(z) > 0$.

Hence all the zeros of $Q(z)$ with $|z| > 1$ lie in
$|z| \leq |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1, already satisfy the above inequality, since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality. This completes the proof of the theorem.

Corollary 6 : Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $k \geq 1$,
$a_0 \leq a_1 \leq \ldots \leq k a_m \geq a_{m+1} \geq \ldots a_{n-1} \geq a_n$
$(n-i)a_i \leq a_{i+1}$, $i = 0, 1, 2, \ldots m-1$
$ja_j \leq (j-1)a_{j-1}$, $j = m+1, \ldots n$.
then for a real $\alpha \neq -a_{n-1}/na_n$ the polar derivative of $P(z)$ with respect to α has exactly n-1 zeros and they lie in
$|z| \leq |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)$
Indeed, if real $\alpha \neq -a_{n-1}/na_n$ then $D_\alpha P(z)$ is of n-1 degree so it has n-1.

Corollary 7 : Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $k \geq 1$,
$a_0 \leq a_1 \leq \ldots \leq k a_m \geq a_{m+1} \geq \ldots a_{n-1} \geq a_n$
$(n-i)a_i \leq a_{i+1}$, $i = 0, 1, 2, \ldots m-1$
$ja_j \geq (j-1)a_{j-1}$, $j = m+1, \ldots n$.
then for a real α the polar derivatives of $P(z)$ with respect to α has atmost n-1 zeros and they lie in
$|z| \leq |a_{n-1}+\alpha n a_n|^{-1}(-a_{n-1} - \alpha n a_n + (n-m)a_m + \alpha(m+1)a_{m+1} + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + \alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)$
The above corollary is obtained by taking k=1.

Theorem 8: Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that $k \geq 1$,
$a_0 \leq a_1 \leq \ldots \leq k a_m \geq a_{m+1} \geq \ldots a_{p-1} \geq a_p$
$(n-i)a_i \leq a_{i+1}$, $i = 0, 1, 2, \ldots m-1$
$ja_j \leq (j-1)a_{j-1}$, $j = m+1, \ldots p$
then for a real $\alpha = -a_{n-1}/na_n = \ldots = -(n-m-1)a_{p+1}/(p+2)a_{p+2}$, where $p = m+1, \ldots n$ the polar derivative of $P(z)$ with respect to α has p zeros and they lie in
$|z| \leq |(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1} + k(n-m)a_m + \alpha(m+1)a_{m+1} + (n-2m)(k-1)|a_m| + |(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m| + (n-m+1)a_{m-1} + k\alpha m a_m - na_0 - \alpha a_1 + |na_0 + \alpha a_1|)\}$

Proof: Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ be a polynomial of degree n.
Then the polar derivative of $P(z)$ is given by $D_\alpha P(z) = n P(z) + (\alpha - z) P'(z)$. Then
$D_\alpha P(z) == [na_0 + \alpha a_1] + [(n-1)a_1 + 2\alpha a_2] z + [(n-2)a_2 + 3\alpha a_3]z^2 + \ldots$
$+ [(n-m+1)a_{m-1} + \alpha m a_m]z^{m-1} + [(n-m)a_m + \alpha(m+1)a_{m+1}]z^m + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2}]z^{m+1} + \ldots$
$+ [2a_{n-2} + \alpha(n-1)a_{n-1}]z^{n-2} + [a_{n-1} + \alpha n a_n]z^{n-1}$.
As $\alpha = -a_{n-1}/na_n = \ldots = -(n-m-1)a_{p+1}/(p+2)a_{p+2}$

$D_\alpha P(z) = [(n-p)a_p + \alpha(p+1)a_{p+1}]z^p + [(n-p+1)a_{p-1} + \alpha p a_p]z^{p-1} + [(n-p+2)a_{p-2} + \alpha(p-1)a_{p-1}] + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3]z^2 + [(n-1)a_1 + 2\alpha a_2] z + [na_0 + \alpha a_1]$
Now consider the polynomial $Q(z) = (1-z) D_\alpha P(z)$ so that
$Q(z) = -[(n-p)a_p + \alpha(p+1)a_{p+1}]z^{p+1} + [(n-p)a_p + \alpha(p+1)a_{p+1} - (n-p+1)a_{p-1} - \alpha p a_p]z^p + \ldots + [(n-m-1)a_{m+1} + \alpha(m+2)a_{m+2} - (n-m)a_m - \alpha(m+1)a_{m+1}]z^{m+1}$
$+ [(n-m)a_m + \alpha(m+1)a_{m+1} - (n-m+1)a_{m-1} - \alpha m a_m]z^m + [(n-m+1)a_{m-1}$

$+ \alpha m a_m - (n-m+2)a_{m-2} - \alpha(m-1)a_{m-1}]z^{m-1} + \ldots$
$+ [(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2]z^2 + [(n-1)a_1 + 2\alpha a_2 - na_0 - \alpha a_1]z$
$+ [na_0 + \alpha a_1]$.
Now if $|z|>1$ then $|z|^{-(n-i)} < 1$ for $i = n-1,\ldots,n-p$
Further,
$|Q(z)| \geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^{p+1} - \{|(n-p)a_p + \alpha(p+1)a_{p+1}$
$- (n-p+1)a_{p-1} - \alpha p a_p|\,|z|^p + |(n-p+1)a_{p-1} - \alpha p a_p -(n-p+2)a_{p-2} -\alpha(p-1)a_{p-1}|\,|z|^{p-1} + \ldots +|(n-m-1)a_{m+1}+\alpha(m+2)a_{m+2}-(n-m)a_m-\alpha(m+1)a_{m+1}||z|^{m+1}$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m||z|^m$
$+|(n-m+1)a_{m-1}+\alpha m a_m-(n-m+2)a_{m-2}-\alpha(m-1)a_{m-1}||z|^{m-1}+\ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|\,|z|^2 + |(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1|\,|z|$
$+|na_0 + \alpha a_1|\}$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{|(n-p)a_p + \alpha(p+1)a_{p+1}$
$- (n-p+1)a_{p-1} - \alpha p a_p| + |(n-p+1)a_{p-1} - \alpha p a_p -(n-p+2)a_{p-2} -\alpha(p-1)a_{p-1}||z|^{-1}+\ldots$
$+|(n-m-1)a_{m+1}+\alpha(m+2)a_{m+2}-(n-m)a_m-\alpha(m+1)a_{m+1}||z|^{-(p-m-1)}$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m||z|^{-(p-m)}$
$+|(n-m+1)a_{m-1}+\alpha m a_m-(n-m+2)a_{m-2}-\alpha(m-1)a_{m-1}||z|^{-(p-m+1)}+\ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2||z|^{-(p-2)}$
$+ |(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1|\,|z|^{-(p-1)}+|na_0 + \alpha a_1||z|^{-p}\}]$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{|(n-p)a_p + \alpha(p+1)a_{p+1}$
$- (n-p+1)a_{p-1} - \alpha p a_p| + |(n-p+1)a_{p-1} - \alpha p a_p -(n-p+2)a_{p-2} -\alpha(p-1)a_{p-1}|+\ldots$
$+|(n-m-1)a_{m+1}+\alpha(m+2)a_{m+2}-(n-m)a_m-\alpha(m+1)a_{m+1}|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+|(n-m+1)a_{m-1}+\alpha m a_m-(n-m+2)a_{m-2}-\alpha(m-1)a_{m-1}|+\ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2| + |(n-1)a_1 + 2\alpha a_2 -na_0 - \alpha a_1|+|na_0 + \alpha a_1|$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{|(n-p)a_p + \alpha(p+1)a_{p+1}$
$- (n-p+1)a_{p-1} - \alpha p a_p| + |(n-p+1)a_{p-1} - \alpha p a_p -(n-p+2)a_{p-2} -\alpha(p-1)a_{p-1}|+\ldots$
$+|(n-m-1)a_{m+1}+\alpha(m+2)a_{m+2}-k(n-m)a_m-\alpha(m+1)a_{m+1}+k(n-m)a_m-(n-m)a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+|(n-m+1)a_{m-1}+\alpha k m a_m-(n-m+2)a_{m-2}-\alpha(m-1)a_{m-1}-\alpha k m a_m+\alpha m a_m|+\ldots$
$+|(n-2)a_2 + 3\alpha a_3 - (n-1)a_1 - 2\alpha a_2|+|(n-1)a_1+2\alpha a_2-na_0-\alpha a_1|+|na_0 + \alpha a_1|\}]$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+ (n-p+1)a_{p-1} + \alpha p a_p - (n-p+1)a_{p-1} - \alpha p a_p +(n-p+2)a_{p-2} +\alpha(p-1)a_{p-1}|+\ldots$
$-(n-m-1)a_{m+1}-\alpha(m+2)a_{m+2}+k(n-m)a_m+\alpha(m+1)a_{m+1}+(k-1)(n-m)|a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|-(n-m+2)a_{m-2}-\alpha(m-1)a_{m-1}$
$+(n-m+1)a_{m-1}+\alpha k m a_m+(1-k)m|a_m|+\ldots - (n-1)a_1 - 2\alpha a_2+(n-2)a_2 + 3\alpha a_3$
$-na_0-\alpha a_1+(n-1)a_1+2\alpha a_2+|na_0+\alpha a_1|\})$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+k(n-m)a_m+\alpha(m+1)a_{m+1}+(k-1)(n-m)|a_m|+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n-m+1)a_{m-1}+\alpha k m a_m+(1-k)m|a_m| -na_0-\alpha a_1+|na_0+\alpha a_1|\}]$
$\geq |(n-p)a_p + \alpha(p+1)a_{p+1}||z|^p[|z|-|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+[k(n-m)+k\alpha m]a_m+\alpha(m+1)a_{m+1}+(k-1)(n-2m)|a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n.m+1)a_{m-1}-na_0-\alpha a_1+|na_0+\alpha a_1|\}]$

> 0 if $|z|>|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+[k(n-m)+k\alpha m]a_m+\alpha(m+1)a_{m+1}+(k-1)(n-2m)|a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n.m+1)a_{m-1}-na_0-\alpha a_1+|na_0+\alpha a_1|\}]$

This shows that if $|z||>|(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+[k(n-m)+k\alpha m]a_m+\alpha(m+1)a_{m+1}+(k-1)(n-2m)|a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n.m+1)a_{m-1}-na_0-\alpha a_1+|na_0+\alpha a_1|\}]$,then $Q(z)>0$.
Hence all the zeros of $Q(z)$ with $|z|>1$ lie in
$|z|\leq |(n-p)a_p + \alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p - \alpha(p+1)a_{p+1}$
$+[k(n-m)+k\alpha m]a_m+\alpha(m+1)a_{m+1}+(k-1)(n-2m)|a_m|$
$+|(n-m)a_m+\alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n-m+1)a_{m-1}-na_0-\alpha a_1+|na_0+\alpha a_1|\}]$.
But those zeros of $Q(z)$ whose modulus is less than or equal to 1,already satisfy the above inequality ,since all the zeros of $D_\alpha P(z)$ are also the zeros of $Q(z)$ as they lie in the circle defined by the above inequality.Hence proof of the theorem is complete.

Corollary : 9 Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that
$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots a_{p-1} \geq a_p$
$(n-i)a_i \leq a_{i+1}$, i= 0, 1, 2, . . .m-1
$ja_j \leq (j-1)a_{j-1}$, j= m+1,. . . p
then for a real $\alpha= -a_{n-1}/na_n =\ldots =-(n-m-1)a_{p+1}/(p+2)a_{p+2}$, where p=m+1,….n the polar derivative of P(z) with respect to α has p zeros and they lie in
$|z|\leq|(n-p)a_p+\alpha(p+1)a_{p+1}|^{-1}\{-(n-p)a_p-\alpha(p+1)a_{p+1}+(n-m)a_m+\alpha(m+1)a_{m+1}$
$+ |(n-m)a_m+ \alpha(m+1)a_{m+1}-(n-m+1)a_{m-1}-\alpha m a_m|$
$+(n-m+1)a_{m-1}+\alpha m a_m -na_0-\alpha a_1+|na_0+\alpha a_1|)\}$

The above corollary is obtained by taking k=1 in theorem 8.

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