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A Homotopy Analysis Approach to a Thermoelastic Interactions under the Boundary Condition: Heat Source Varying Exponentially with Time and Zero Stress

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Abstract: ***In this work, we consider a one-dimensional problem for a half-space in the context of the L-S theory of generalized thermo elasticity with one relaxation time parameter. The surface of the half-space is assumed to be traction free and subjected to the effects of a heat source varying exponentially with time at the boundary. The main objective of this present paper is to study the above problem based on the LS theory with the help of Homotopy analysis method proposed by S.J. Liao (1992). See the Appendix for details. The inversion of the Laplace transform solutions are carried out numerically using Bellman method (Bellman et al., 1966) and the obtained results are presented graphically. The effects of the heat source varying with time and zero traction force are studied on temperature, stress and displacement. Also we see the numerical result with the help of graph.***

Keywords: Homotopy, Thermoelastic, Homotopy perturbation method.

1. Introduction

The theories were proposed by Lord and Shulman (1967) (LS model), Green and Lindsay (1972) (GL model) and Green and Naghdi (1991; 1992; 1993) (GN models) based on "second sound" effects, i.e., propagation of heat as a wave like phenomenon.

In 1967, Lord and Shulman attempt to eliminate the paradox of infinite velocity of thermal disturbances inherent in the CTE. This model is based on a modified Fourier's law but in addition a single relaxation time was considered. This theory was extended by Dhaliwal and Sherief (1980a) to include the anisotropic case. The uniqueness of the solution for this theory was proved under different conditions by Ignaczak (1979; 1982), by Dhaliwal and Sherief (1980b) and by Sherief (1987). In the L-S model, finite speed of thermal disturbance has been considered with thermal relaxation time. The heat conduction equation in this model is of hyperbolic type and is closely connected with the theories of second sound.

Green and Lindsay (1972) also proposed a theory of generalized thermoelasticity with two relaxation time parameters and modified both the energy equation and constitutive equations. The GL model admits second sound without violating Fourier's law. Both the theories are structurally different, but one can be obtained as a particular case of the other. Various problems related to the above theories have been investigated by Youssef (2006; 2009), Sherief and Megahed (1999), Sherief and Youssef (2004), Ezzat *et al.* (2001), Lahiri *et al.* (2010a; 2010b).

A recent theoretical development of this subject is due to Green and Naghdi (1991; 1992; 1993). Providing sufficient basic modifications in the governing equations Green and Naghdi developed a new general theory of thermoelasticity that permits treatment of a much wider class of heat flow problems. They divided their theory into three parts and referred as types I, II and III. The linearized version of constitutive equations of GN I (1991) is same as the classical thermoelasticity theory which is based on Fourier's law, whereas the linearized version of GN II (1993) theory permits propagation of thermal waves at finite speed. In the GN-II model, the internal rate of production of entropy is assumed to be identical to zero, i.e., there is no dissipation of thermal energy and this theory is referred to as thermoelasticity without energy dissipation theory. In development of the GN III (1992) model, the constitutive equations are derived by including a thermal displacement gradient in addition to a temperature gradient among the constitutive variables. However, this model admits dissipation of energy in general.

Most of the thermoelasticity (generalized or coupled) problems have been solved by using the potential function. This method is not always suitable as discussed by Anwar and Sherief (1988) and Sherief (1993). Their discussion may be summarized by (I) the boundary and initial conditions for physical problems are directly related to the physical quantities under consideration and not to the potential function and (II) the solution of the physical problem in natural variables is convergent while other potential function representations are not convergent always.

The alternatives to the potential function approach are as follows (I) State-Space approach: This method is essentially an expansion in a series in terms of the coefficient matrix of the field variables in ascending powers and applying Caley-Hamilton theorem, which requires extensive algebra, and (II)

Eigenvalue approach: This method reduces the problem on vector-matrix differential equation to an algebraic eigenvalue problems and the solutions for the field variables are achieved by determining the eigenvalues and the corresponding eigenvectors of the coefficient matrix. In the eigenvalue approach the physical quantities are directly involved in the formulating of the problem and as such the boundary and initial conditions can be applied directly. Body forces and/or heat sources are also accommodated in both the theories, cf. Das *et al*. (1997; 2009), Lahiri *et al*. (2010), Kar and Lahiri (2004) and Sarkar and Lahiri (2012).

Saleh (2005), Youssef (2006; 2009) studied some one-dimensional problems in thermoelasticity (generalized or coupled) including heat sources by the state space approach and generally in their models, heat sources included the *Dirac delta function* and *Heavisides unit step function* and consequently they obtained solutions easily by the state-space approach, but here we are interested in considering the heat source in any form and solve by the *eigenvalue approach* developed in Sarkar and Lahiri (2012).

In this work, we consider a one-dimensional problem for a half-space in the context of the L-S theory of generalized thermoelasticity with one relaxation time parameter. The surface of the half-space is assumed to be traction free and subjected to the effects of a heat source varying exponentially with time at the boundary. The main objective of this present paper is to study the above problem based on the LS theory with the help of Homotopy analysis method proposed by S.J. Liao (1992). See the Appendix for details. The inversion of the Laplace transform solutions are carried out numerically using Bellman method (Bellman *et al*., 1966) and the obtained results are presented graphically. The effects of the heat source varying with time and zero traction force are studied on temperature, stress and displacement. Also we see the numerical result with the help of graph.

2. Governing Equations

For a homogeneous, isotropic elastic solid, the basic equations for the linear generalized theory of thermoelasticity with one relaxation time parameter proposed by Lord and Shulman (1967) in the absence of body forces are:

(I) Equation of motion

$$\sigma_{ij,j}=\rho\frac{\partial^2 u_i}{\partial t^2} \quad (2.1)$$

(II) Heat conduction equation

$$\kappa T_{,ii}=\left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T+\gamma T_0 e\right) \quad (2.2)$$

(III) Stress–displacement–temperature relations

$$\sigma_{ij}=2\mu e_{ij}+\lambda e_{kk}\delta_{ij}-\gamma\left(T-T_0\right)\delta_{ij} \quad (2.3)$$

where *i, j=1,2,3* refer to general coordinates.

3. Formulation of the Problem

We consider a homogeneous isotropic and thermoelastic half-space which fills the region subjected to a heat source varying exponentially with time on the boundary plane and the surface $x=0$ is assume to be traction free. The governing equation will be written in the context of the theory of thermoelasticity with one relaxation time parameter (LS model), when the body has no body forces. Clearly, this is a one dimensional problem and all the state functions depend only on the space variable x and time variable t. The displacement component of this case are of the form

$$u_i=\left(u,0,0\right) \qquad u_y=u_z=0$$

We will use the Cartesian co-ordinates and the components of displacements $u_i=\left(u,0,0\right)$ and write them as follows

$$\frac{\partial^2 u}{\partial t^2}=\left(\lambda+2\mu\right)\frac{\partial^2 u}{\partial x^2}-\gamma\frac{\partial T}{\partial x} \quad (3.1)$$

$$\kappa\frac{\partial^2 T}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T+\gamma T_0 e\right) \quad (3.2)$$

$$\sigma_{xx}=\left(\lambda+2\mu\right)\frac{\partial u}{\partial x}-\gamma\left(T-T_0\right) \quad (3.3)$$

where $e=\dfrac{\partial u}{\partial x}$

To transform the above equations in non-dimensional forms, we define the following non-dimensional variables

$$x'=c_1\eta x,\ \ u'=c_1\eta u,\ \ t'=c_1^2\eta t,\ \ \tau'=c_1^2\eta\tau,\ \ \theta'=\frac{\gamma\left(T-T_0\right)}{\lambda+2\mu}$$

$$\sigma'_{xx}=\frac{\sigma_{xx}}{\lambda+2\mu},\ \ c_1^2=\frac{\lambda+2\mu}{\rho},\ \ \eta=\frac{\rho C_E}{\kappa}$$

Equations (3.1)-(3.3) then reduce to the following non-dimensional forms (the primes are dropped for convenience)

$$\frac{\partial^2 u}{\partial x^2}-a_1\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2} \quad (4)$$

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}+\tau\frac{\partial^2\theta}{\partial t^2}+a_2\left[\frac{\partial^2 u}{\partial t\partial x}+\tau\frac{\partial^3 u}{\partial t^2\partial x}\right] \quad (5)$$

$$\sigma_{xx}=\frac{\partial u}{\partial x}-a_1\theta \quad (6)$$

Where, $a_1=\dfrac{\gamma T_0}{\lambda+2\mu}, a_2=\dfrac{\gamma}{\rho c_v}$.

Solution using the Homotopy perturbation method:

From (4) and (6), we get,

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}=\frac{\partial^2\sigma_{xx}}{\partial t^2}+a_1\frac{\partial^2\theta}{\partial t^2} \quad (7)$$

Using equation (5) and (6), we get,

$$\frac{\partial^2\theta}{\partial x^2}=\left(1+\varepsilon\right)\frac{\partial\theta}{\partial t}+\tau\left(1+\varepsilon\right)\frac{\partial^2\theta}{\partial t^2}+a_2\left[\frac{\partial\sigma_{xx}}{\partial t}+\tau\frac{\partial^2\sigma_{xx}}{\partial t^2}\right] \quad (8)$$

Where, $\varepsilon=a_1a_2$.

Equations (7) and (8) represents the second order coupled thermoelastic equations in temperature θ and stress σ_{xx}.

Let us assume boundary conditions as follows,

$$\left.\begin{aligned}\theta(t,0)=e^{-t}\\ \sigma_{xx}(t,0)=0\end{aligned}\right\} \quad (9)$$

According to HPM, we construct the following simple homotopies:

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}+p\left[-\frac{\partial^2\sigma_{xx}}{\partial t^2}-a_1\frac{\partial^2\theta}{\partial t^2}\right]=0 \quad (10)$$

$$\frac{\partial^2\theta}{\partial x^2}+p\left[-(1+\varepsilon)\frac{\partial\theta}{\partial t}-\tau(1+\varepsilon)\frac{\partial^2\theta}{\partial t^2}-a_2\frac{\partial\sigma_{xx}}{\partial t}-a_2\tau\frac{\partial^2\sigma_{xx}}{\partial t^2}\right]=0 \quad (11)$$

Where $p\in[0,1]$ is an embedding parameter, we use it to expand the solution in the following form:

$$\sigma_{xx}(x,t)=\sigma_{xx}^0+p\sigma_{xx}^1+p^2\sigma_{xx}^2+\ldots \quad (12)$$

$$\theta(x,t)=\theta^0+p\theta^1+p^2\theta^2+p^3\theta^3+\ldots \quad (13)$$

Where, in σ_{xx}^i , i denotes superscript notation not power.
The approximate solution can be obtained by setting $p=1$ in equation (12) and (13).

Now, substituting $\sigma_{xx}(x,t)$ and $\theta(x,t)$ from (12), (13) to (10), (11) respectively, and equating the terms with identical powers of p, we can obtained series of linear equations. These linear equations are easy to solve by using Mathematica software to get as many equations as we need in the calculation of the numerical as well as explicit solutions. Here we write only the first few linear equations :

$$p^0:\quad \frac{\partial^2\sigma_{xx}^0}{\partial x^2}=0 \quad (14a)$$

$$\frac{\partial^2\theta^0}{\partial x^2}=0 \quad (14b)$$

$$p^1:\quad \frac{\partial^2\sigma_{xx}^1}{\partial x^2}=\frac{\partial^2\sigma_{xx}^0}{\partial t^2}+a_1\frac{\partial^2\theta^0}{\partial t^2} \quad (15a)$$

$$\frac{\partial^2\theta^1}{\partial x^2}=(1+\varepsilon)\frac{\partial\theta^0}{\partial t}+\tau(1+\varepsilon)\frac{\partial^2\theta^0}{\partial t^2}+a_2\frac{\partial\sigma_{xx}^0}{\partial t}+a_2\tau\frac{\partial^2\sigma_{xx}^0}{\partial t^2} \quad (15b)$$

$$p^2:\quad \frac{\partial^2\sigma_{xx}^2}{\partial x^2}=\frac{\partial^2\sigma_{xx}^1}{\partial t^2}+a_1\frac{\partial^2\theta^1}{\partial t^2} \quad (16a)$$

$$\frac{\partial^2\theta^2}{\partial x^2}=(1+\varepsilon)\frac{\partial\theta^1}{\partial t}+\tau(1+\varepsilon)\frac{\partial^2\theta^1}{\partial t^2}+a_2\frac{\partial\sigma_{xx}^1}{\partial t}+a_2\tau\frac{\partial^2\sigma_{xx}^1}{\partial t^2} \quad (16b)$$

The solution of equation (14) can be calculated by using the boundary conditions (9):

$$\sigma_{xx}^0(x,t)=\sigma_{xx}(t,0)=0,\qquad \theta^0(x,t)=\theta(t,0)=e^{-t} \quad (17)$$

Then, we can derive the solution of (15) in the following form,

$$\sigma_{xx}^1(x,t)=\int_0^x\int_0^x\frac{\partial^2\sigma_{xx}^0}{\partial t^2}dt\ dx+a_1\int_0^x\int_0^x\frac{\partial^2\theta^0}{\partial t^2}dt\ dx$$

$$\sigma_{xx}^1=a_1\left(\frac{x^2}{2!}\right)e^{-t} \quad (18)$$

$$\theta^1(x,t)=\iint(1+\varepsilon)\frac{\partial\theta^0}{\partial t}dxdx+\iint\tau(1+\varepsilon)\frac{\partial^2\theta^0}{\partial t^2}dxdx+\iint a_2\frac{\partial\sigma_{xx}^0}{\partial t}dxdx+\iint a_2\tau\frac{\partial^2\sigma_{xx}^0}{\partial t^2}dxdx$$

$$\theta^1(x,t)=(1+\varepsilon)(\tau-1)\left(\frac{x^2}{2!}\right)e^{-t} \quad (19)$$

Similarly,

$$\sigma_{xx}^2(x,t)=a_1\left[\tau(1+\varepsilon)-\varepsilon\right]\frac{x^4}{4!}e^{-t} \quad (20)$$

$$\theta^2(x,t)=(1-\tau)[(1+\varepsilon)^2-\tau(1+\varepsilon)^2-1]\frac{x^4}{4!}e^{-t} \quad (21)$$

$$\sigma_{xx}^2(x,t)=a_1[\tau(1+\varepsilon)+(1+\varepsilon)^2-2\tau(1+\varepsilon)^2-\varepsilon(2-\tau)+\tau^2(1+\varepsilon)^2]\frac{x^6}{6!}e^{-t} \quad (22)$$

Therefore the complete approximate solution can be readily obtained by the same iterated process using the Mathematica package.

Truncate the series solution obtained by HPM:

$$\sigma_{xx}(x,t)\cong\tilde{\sigma}_{xx}(x,t)=\sum_{i=0}^{3}\sigma_{xx}^i(x,t)=\left[m_1\frac{x^2}{2!}+m_2\frac{x^4}{4!}+m_3\frac{x^6}{6!}\right]e^{-t} \quad (24)$$

$$\theta(x,t)\cong\tilde{\theta}(x,t)=\sum_{i=0}^{3}\theta^i(x,t)=[1+n_1\frac{x^2}{2!}+n_2\frac{x^4}{4!}+n_3\frac{x^6}{6!}]e^{-t} \quad (25)$$

From equation (6), we find that

$$\frac{\partial u}{\partial x}=[a_1+(m_1+a_1n_1)\frac{x^2}{2!}+(m_2+a_1n_2)\frac{x^4}{4!}+(m_3+a_1n_3)\frac{x^6}{6!}]e^{-t}$$

Now, integrating above equation with respect to x, we get

$$u(x,t)=[\frac{a_1x}{1!}+(m_1+a_1n_1)\frac{x^3}{3!}+(m_2+a_1n_2)\frac{x^5}{5!}+(m_3+a_1n_3)\frac{x^7}{7!}]e^{-t} \quad (26)$$

Where,

$m_1=a_1,\quad m_2=a_1[\tau(1+\varepsilon)-\varepsilon]$

$m_3=a_1[\tau(1+\varepsilon)+(1+\varepsilon)^2-2\tau(1+\varepsilon)^2-2\varepsilon+\tau\varepsilon+\tau^2(1+\varepsilon)^2]$

$n_1=(\tau-1)(1+\varepsilon)$

$n_2=(1-\tau)[(1+\varepsilon)^2-1-\tau(1+\varepsilon)^2]$

$n_3=[-(1+\varepsilon)^3+3\tau(1+\varepsilon)^3+2\varepsilon(1+\varepsilon)-3\tau^2(1+\varepsilon)^3-4\tau\varepsilon(1+\varepsilon)+\tau^3(1+\varepsilon)^3+2\tau^2\varepsilon(1+\varepsilon)-\varepsilon+\tau\varepsilon]$

4. Numerical Results

In this section, we illustrate the problem with numerical values of the field variables like displacement, temperature and stress for a material in space-time domain. Therefore, with the help of Mathematica software, we compute values of the variables by using solutions obtained in (24)-(26) and employing the numerical method. We choose the copper material for this purpose and physical data for it are given as follows:

$$\lambda = 7.76\times10^{10}\,Nm^{-2}, \mu = 3.86\times10^{10}\,Nm^{-2},$$

$$\alpha_t = 1.78\times10^{-5}\,K^{-1}, \kappa = 8886.73\,\mathrm{sm}^{-2},$$

$$c_v = 383.1\,JKg^{-1}K^{-1}, \rho = 8954\,Kgm^{-3}.$$

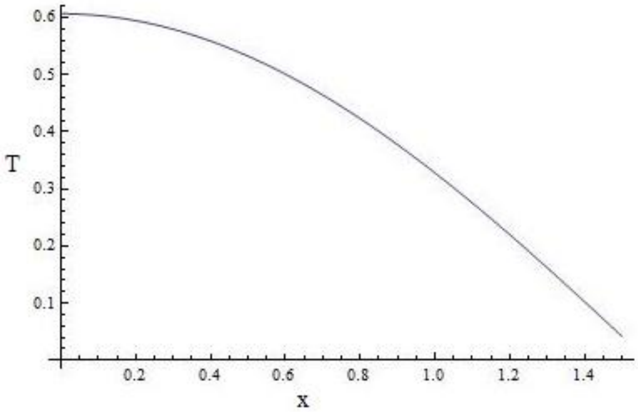

Figure 1: Variation of temperature, T vs. *x* at *t*=0.2.

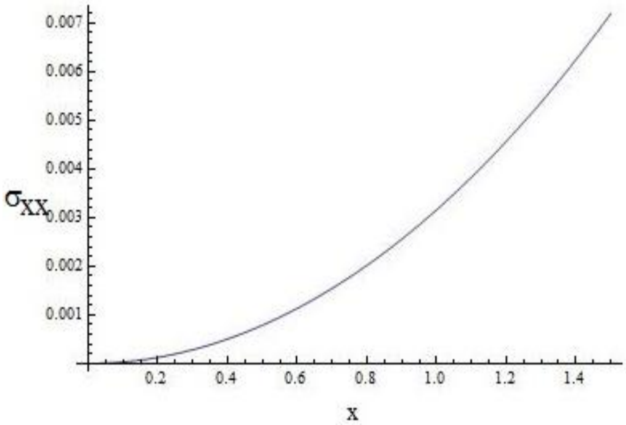

Figure 2: Variation of Stress, σ_{xx} vs. *x* at t=0.2.

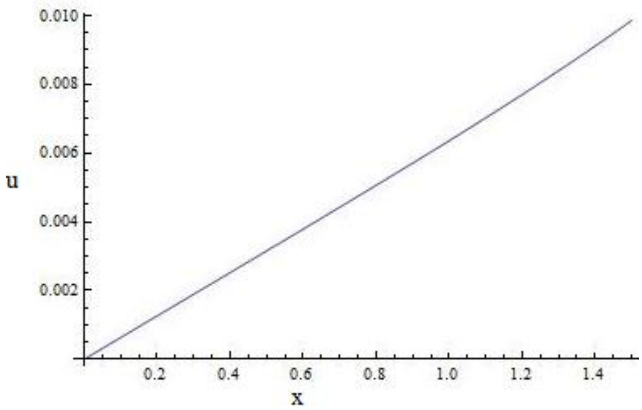

Figure 3: Variation of displacement, *u* vs. *x* at *t*=0.2.

Figs. (1, 2, 3) shows the variation of temperature, stress and displacement with respect to distance, *x* under thermoelasticity LS model at time t=0.2. Fig. 1 indicates that when the boundary of the half space is subjected to a heat source varying exponentially with time and zero stress, the temperature decreases gradually and finally gets zero value after travelling a distance. Fig. 2 shows that stress increases with space variable *x*. Finally fig. 3 shows that displacement increases almost linearly with *x*.

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