A Realistic Study of the Rayleigh-Bénard Problem in the Newtonian Nanofluids with a Uniform Heat Source: Free-Free Case

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Abstract: The main purpose in this investigation is to use the Buongiorno’s mathematical model for studying the effect of an internal heat source which produces a constant volumetric heat on the onset of convective instability in a confined medium, filled of a Newtonian nanofluid layer and heated from below, this layer is assumed to have a low concentration of nanoparticles. The linear study in the free-free case shows that the thermal stability depends of the volumetric heat delivered by the internal source, the Brownian motion, the thermophoresis of the nanoparticles and other thermo-physical properties of nanoparticles. The studied problem will be solved analytically by converting our boundary value problem to an initial value problem, after this step we will approach the searched solutions with polynomials of high degree.

Keywords: linear stability, nanofluid, Buongiorno’s model, internal heat source, free-free case.

1. Introduction

For increasing the effective thermal conductivity of the coolant fluids in a confined geometry, we prefer using fluids containing nano-sized metallic particles (about 1-100nm) in suspension to obtain a nanofluid characterized by a high effective thermal conductivity compared to the regular fluids (water - oil - ethylene glycol). The experiment shows that the presence of the nanoparticles in a base fluid allows us to obtain a significant growth in the thermal conductivity of the mixture (base fluid + nanoparticles), for this purpose we find that the nanofluids are currently used in the cooling of advanced electronic or nuclear systems.

The nanofluid term was introduced by Choi [1] in 1995 and remains usually used to characterize this type of colloidal suspension. Buongiorno [2] was the first researcher who treated the convective transport problem in nanofluids, he was established the conservation equations of a non-homogeneous equilibrium model of nanofluids for mass, momentum and heat transport. The thermal problem of instability in nanofluids with rigid-free and free-free boundaries was studied by Tzou [3,4] using the eigenfunction expansions method. The onset of convection in a horizontal nanofluid layer of finite depth was studied by Nield and Kuznetsov [5], they found that the critical Rayleigh number can be decreased or increased by a significant quantity depending on the relative distribution of nanoparticles between the top and bottom walls.

In this paper, we will study the Rayleigh-Bénard problem for the Newtonian nanofluids with a uniform heat source in the free-free case using a new type of boundary conditions for the nanoparticles which assumes the nanoparticle flux must be zero on the impermeable boundaries. D.A. Nield and A.V. Kuznetsov [6] are considered as the first ones who used this type of boundary conditions for the nanoparticles.

The new model of boundary conditions for the nanoparticles is physically more realistic since it combines the contribution of the Brownian motion and the thermophoresis of the nanoparticles, for this reason we find currently several authors [7-11] are using this type of model to study the natural convection in nanofluids.

To show the accuracy of our method in this study, we will check some results treated by D.Yadav et al. [12] concerning the study of the convective instability of regular fluids in presence of an internal heat source which produces a constant volumetric heat in the free-free case.

2. Mathematical Formulation

We consider an infinite horizontal layer of an incompressible Newtonian nanofluid characterized by a low concentration of nanoparticles, heated uniformly from below and confined between two identical horizontal surfaces where the temperature is constant and the nanoparticle flux is zero on the boundaries, this layer will be subjected to an internal heat source which will provide a constant volumetric heat $Q_s$ and also to the gravity field $g$ (see Figure 1). The thermo-physical properties of nanofluid (viscosity, thermal conductivity, specific heat) are assumed constant in the analytical formulation except for the density variation in the momentum equation which is based on the Boussinesq approximations.

![Figure 1: Physical configuration](image)
Within the framework of the assumptions which were made by Buongiorno [2] and Tzou [3,4] in their publications for the Newtonian nanofluids, we can write the basic equations of conservation which govern our problem in dimensionless form as follows:

\[ \vec{V}_t + \vec{V} \cdot \nabla \vec{V} = 0 \]  
(1)

\[ \rho_t \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + \eta \nabla^2 \vec{V} \]  
(2)

\[ + \left[ \rho_b [1 - \beta (T^* - T)] \right] (1 - \chi^*) + \rho p \chi^* \right] \]  
and

\[ (pc)_{T} \left[ \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T \right] = \kappa \nabla^2 T + Q_z \]  
(3)

\[ + (pc)_{p} \left[ D_{B} \nabla^2 \chi - \nabla^2 \vec{V} + \left( \frac{\nabla}{T} \right) \nabla \vec{T} \cdot \nabla \vec{T} \right] \]  
(4)

where \( \rho_t \) is the density of the base fluid, \( \rho_b \) is the fluid density at reference temperature \( T_r \), \( \rho_p \) is the nanoparticle density, \( \beta \) is the thermal expansion coefficient of the base fluid, \( \vec{V} \) is the velocity vector, \( t^* \) is the time, \( P^* \) is the pressure, \( T^* \) is the temperature, \( \chi^* \) is the volume fraction of nanoparticles, \( \eta \) is the viscosity of the nanofluid, \( \kappa \) is the thermal conductivity of the nanofluid, \( D_{B} \) is the Brownian diffusion coefficient, \( D_{T} \) is the thermophoretic diffusion coefficient, \( (pc)_{T} \) is the heat capacity of the base fluid, \( (pc)_{p} \) is the heat capacity of the nanoparticle, \((x^*, y^*, z^*)\) are the cartesian coordinates, \( \vec{V} \) is the vector differential operator.

In this study the asterisks are used to distinguish the dimensionless variables from the nondimensional variables (without asterisks).

If we consider the following dimensionless variables:

\[ (x^*; y^*; z^*) = (xh; yh; zh); \quad t^* = \frac{ht}{h^2}; \quad \vec{V}^* = \frac{h}{h} \vec{V} \]

Then, we can get from equations (1)-(4) the following adimensional forms:

\[ \vec{V}_t + \vec{V} \cdot \nabla \vec{V} = 0 \]  
(5)

\[ \rho_t^{-1} \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + (R_s \eta) z + \vec{V} \]  
(6)

\[ + [(1 - \chi^*) T - R_N \chi^*] \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T = \kappa \]  
(7)

\[ + N_{A_b} \chi^*_b \nabla T + N_{A_b} \chi^*_b \nabla T \]  
(8)

Where \( \rho_t = \eta / \rho_a \) is the Prandtl number, \( \nu_a = \alpha / \nu_a \) is the Lewis number, \( H_s = Q \beta / k(T_r - T_c) \) is the dimensionless constant heat source strength, \( R_s = \rho_s g B(T_r - T_c)/\eta_a \) is the thermal Rayleigh number, \( R_{N} = \rho_p (1 - \chi^*) + \rho \chi^* g \beta h / \eta_a \) is the basic density Rayleigh number, \( R_{N} = \rho_p (1 - \chi^*) + \rho \chi^* g \beta h / \eta_a \) is the concentration Rayleigh number, \( N_{A_b} = D_B T_r / D_{T} \chi^*_b \) is the modified diffusivity ratio, \( N_{B} = (pc)_p \chi^*_b / (pc) \) is the modified particle-density increment, \( a = \kappa / (pc) \) is the thermal diffusivity of the nanofluid, \( \chi^*_b \) is the reference value for the nanoparticle volume fraction.

### 2.1 Basic Solution

The basic solution of our problem is a quiescent thermal equilibrium state, it’s assumed to be independent of time where the equilibrium variables are varying in the z-direction only, therefore:

\[ \vec{V}_b = 0 \]  
(9)

\[ T_b = 1 \]  
(10)

\[ T_b = 0 \]  
(11)

If we introduce the precedent results into equations (6)-(8), we obtain:

\[ \vec{V}(P_b + R_N \chi^*) = [(1 - \chi^*_b) T - R_N \chi^*] \nabla \vec{V} + (\vec{V} \cdot \nabla) T = \kappa \]  
(12)

\[ + N_{A_b} \chi^*_b \nabla T + N_{A_b} \chi^*_b \nabla T \]  
(13)

\[ + N_{A_b} \chi^*_b \nabla T + N_{A_b} \chi^*_b \nabla T \]  
(14)

After using the boundary conditions (10) and (11), we can integrate the equation (14) between 0 and \( z \) for obtaining:

\[ \chi_b = N_A (1 - T_b) + \chi_b \]  
(15)

Where \( \chi_0 = (\chi^* - \chi^*_b) / \chi^*_b \) is the relative nanoparticle volume fraction at \( z = 0 \).

If we take into account the expression (15), we can get after simplification of the equation (13):

\[ \frac{d^2 T_b}{dz^2} = -H_s \]  
(16)

Finally, we obtain after an integrating of the equation (16) between 0 and 1:

\[ T_b = - \frac{1}{2} H_s z^2 + \frac{1}{2} H_s - 1 \]  
(17)

\[ \chi_b = \frac{1}{2} N_A H_s z^2 - N_A (\frac{1}{2} H_s - 1) \]  
(18)

### 2.2 Stability Analysis

For analyzing the stability of the system, we superimpose infinitesimal perturbations on the basic solutions as follows

\[ T = T_b + T' \; \vec{V} = \vec{V}_b + \vec{V}' ; \; P = P_b + P' ; \; \chi = \chi_b + \chi' \]  
(19)

In the framework of the Boussinesq approximations, we can neglect the terms coming from the product of the temperature and the volumetric fraction of nanoparticles in equation (6), if we suppose also that we are in the case of small temperature gradients in a dilute suspension of nanoparticles, we can obtain after introducing the expressions (19) into equations (5)-(8) the following linearized equations:

\[ \vec{V}_t + \vec{V} \cdot \nabla \vec{V} = 0 \]  
(20)

\[ \rho_t^{-1} \left[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\nabla p + (R_s \eta) z + \vec{V} \]  
(21)

\[ + [(1 - \chi^*) T - R_N \chi^*] \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T = \kappa \]  
(22)

\[ + N_{A_b} \chi^*_b \nabla T + N_{A_b} \chi^*_b \nabla T \]  
(23)

Where \( f_1 = DT_b \), \( f_2 = N_{A_b} \chi^*_b D(\chi_b + 2N_A T_b) \), \( f_3 = N_{A_b} \chi^*_b^2 DT_b \), \( f_4 = DT_b \) and \( D = d/dz \).
After application of the curl operator twice to equation (21) and using the equation (20), we obtain the following z-component of the momentum equation:

$$\nabla^2 w = \nabla^2 w' + R_4 \nabla^2 T' - R_N \nabla^2 x'$$  \hspace{1cm} (24)

Where $\nabla^2 = (\frac{\partial^2}{\partial x^2}) + (\frac{\partial^2}{\partial y^2})$ is the two-dimensional Laplacian operator on the horizontal plane.

Analyzing the disturbances into normal modes, we can simplify the equations (22) - (24) by assuming that the perturbation quantities are of the form:

$$(w', T', x') = (w(z), T(z), x(z)) e^{i(k_x x + k_y y) + \sigma t}$$  \hspace{1cm} (25)

After introducing the expressions (25) into equations (22) - (24), we obtain:

$$\rho \frac{\partial^2 w}{\partial^2} = (D^2 - k^2) w - k^2 R_4 T + k^2 R_0 X$$  \hspace{1cm} (26)

$$\sigma T' + f_1 w' = (D^2 - k^2) T' + f_2 D T' + f_3 D X$$  \hspace{1cm} (27)

$$\sigma X + f_4 w' = N_A L_z^{-1}(D^2 - k^2) T' + L_z^{-1}(D^2 - k^2) X$$  \hspace{1cm} (28)

Where $\sigma$ is the dimensionless growth rate, $k_x$ and $k_y$ are respectively the dimensionless wave numbers along the $x$ and $y$ directions, $k = \sqrt{k_x^2 + k_y^2}$ is the resultant dimensionless wave number.

In the free-free case, the equations (26) - (28) will be solved subject to the following boundary conditions:

$$\omega = D^2 \omega = T = D(X + N_A T) = 0 \hspace{1cm} at \hspace{0.5cm} z = 0; 1$$  \hspace{1cm} (29)

### 2.3 Method of Solution

In this study we assume that the principle of exchange of stability is valid. As we are interested in a stationary stability study characterized by $\sigma = 0$, then the equations (26)-(28) become:

$$(D^2 - k^2)^2 \omega = -k^2 R_4 T + k^2 R_0 X = 0$$  \hspace{1cm} (30)

$$f_1 w' - (D^2 - k^2) T' - f_2 D T' - f_3 D X = 0$$  \hspace{1cm} (31)

$$f_4 w' - N_A L_z^{-1}(D^2 - k^2) T' - L_z^{-1}(D^2 - k^2) X = 0$$  \hspace{1cm} (32)

We can solve the equations (30) - (32) which are subjected to the conditions (29), by making a suitable change of variables that makes the number of variables equal to the number of boundary conditions, to obtain a set of eight first order ordinary differential equations which we can write in the following form:

$$\frac{d}{dz} u_i(z) = a_{ij} u_j(z) \hspace{1cm} 1 \leq i,j \leq 8$$  \hspace{1cm} (33)

With:

$$a_{ij} = a_{ij}(z, k, R_4, H_4, N_B, L_e, R_N, N_A)$$

The solution of the system (33) in matrix notation can be written as follows:

$$U = BC$$  \hspace{1cm} (34)

Where $B = \left( b_i(z) \right)_{1 \leq i \leq 8}$ is a square matrix of order $8 \times 8$, $U = \left( u_i(z) \right)_{1 \leq i \leq 8}$ is the unknown vector column of our problem, $C = \left( c_i \right)_{1 \leq i \leq 8}$ is a constant vector column.

If we assume that the matrix $B$ is written in the following form:

$$B = \left( \begin{array}{cccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)_{k \leq 8}$$

Therefore, the use of four boundary conditions at $z = 0$, allows us to write each variable $u_i(z)$ as a linear combination only for four functions $u_i(z)$, such that:

$$b_{ij}(0) = u_j(0) = \delta_{ij}$$  \hspace{1cm} (36)

After introducing the new expressions of the variables $u_i(z)$ in the system (33), we will obtain the following equations:

$$\frac{d}{dz} u_i(z) = a_{ij} u_j(z) \hspace{1cm} 1 \leq i,j \leq 8$$  \hspace{1cm} (37)

For each value of $j$, we must solve a set of eight first order ordinary differential equations which are subjected to the initial conditions (36), by approaching these variables with real power series defined in the interval $[0,1]$ and truncated at the order $N$, such that:

$$u_i(z) = \sum_{p=0}^{N} d^i_{p} z^p$$  \hspace{1cm} (38)

A linear combination of the solutions $u_i(z)$ satisfying the boundary conditions (29) at $z = 1$ leads to a homogeneous algebraic system for the coefficients of the combination. A necessary condition for the existence of nontrivial solution is the vanishing of the determinant which can be formally written as:

$$f(R_a, k, H_s, N_B, L_e, R_N, N_A) = 0$$  \hspace{1cm} (39)

If we give to each control parameter $(H_s, N_B, L_e, R_N, N_A)$ its value, we can plot the neutral curve of the stationary convection by the numerical research of the smallest real positive value of the thermal Rayleigh number $R_a$ which corresponds to a fixed wave number $k$ and verifies the dispersion relation (39). After that, we will find a set of points $(k, R_a)$ which help us to plot our curve and find the critical value ($k_c, R_{ac}$) which characterizes the onset of the convective stationary instability, this critical value represents the minimum value of the obtained curve.

### 2.4 Validation of the Method

The truncation order $N$ which correspond at the convergence of our method is determined, when the five digits after the comma of the critical thermal Rayleigh number $R_{ac}$ remain unchanged (see Table 1).

To validate our method, we compared our results with those obtained by Dhananjay Yadav et al. [12] concerning the Rayleigh-Bénard problem for the regular fluids in the presence of an internal heat source. To make this careful comparison, we must take into consideration the restrictions: $L_e^{-1} = R_N = N_A = N_B = 0$ in the governing equations of our problem (see Table 2).

According to the below results, we notice that there is a very good agreement between our results and the previous works, hence the accuracy of the used method. Briefly, the convergence of the results depends greatly on the truncation order $N$ of the power series and also of the heat source strength $H_s$. Finally, to ensure the accuracy of our obtained critical values for the studied Newtonian nanofluids, we will take as truncation order: $N = 32$
Table 1: The exact stationary instability threshold of a Newtonian nanofluid for $H_s = 20$ and $H_s = 60$. 

<table>
<thead>
<tr>
<th>N</th>
<th>$k_c$</th>
<th>$R_{ac}$</th>
<th>$k_c$</th>
<th>$R_{ac}$</th>
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<td>2.72348</td>
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<td>32</td>
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<td>35</td>
<td>2.47232</td>
<td>449.83936</td>
<td>2.72348</td>
<td>209.34745</td>
</tr>
</tbody>
</table>

Exact value $2.47232 \times 10^{-6}$ $2.72348 \times 10^{-6}$ $2.0934745$

Table 2: The obtained stationary instability threshold by D. Yadav et al. [12] and us, for the regular fluids for various values of $H_s$.

<table>
<thead>
<tr>
<th>$H_s$</th>
<th>$k_c$</th>
<th>$R_{ac}$</th>
<th>$k_c$</th>
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<td>232.11473</td>
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<td></td>
</tr>
</tbody>
</table>

3. Results and Conclusion

To study the effect of a parameter $(H_s, N_B, L_e, R_N, N_A)$ on the onset of the convective instability for the Newtonian nanofluids in the presence of an internal heat source which produces a constant volumetric heat, we must determine the variation of the critical thermal Rayleigh number $R_{ac}$ as a function of the heat source strength $H_s$ for different values of this parameter (see Figure 2 - Figure 5).

Figure 2: Plot of $R_{ac}$ as a function of $H_s$ and $N_B$ for $L_e = 100$, $R_N = 1$, $N_A = 0.1$.

In this paper, we have examined the effect of a uniform heat source on the onset of convection in a Newtonian nanofluid layer heated uniformly from below in the case where the nanoparticle flux is zero on the impermeable boundaries, this study shows that the modified particle-density increment $N_B$ has a negligible effect on the onset of the convective instability in the Newtonian nanofluids, therefore the contribution of Brownian motion and thermophoresis in the thermal energy equation (7) can be neglected. Rather, the Brownian motion and thermophoresis directly enter in the expression (8) the conservation of nanoparticles.
The precedent Figures show also that an increase either in the heat source strength $H_s$, in the Lewis number $L_e$, in the concentration Rayleigh number $R_N$ or in the modified diffusivity ratio $N_A$ allows us to accelerate the onset of the convection, hence they have a destabilizing effect.

The obtained results may be summarized as follows:

- The heat source strength $H_s$ increases the energy supply to the system, and hence increases the driving force which accelerates the onset of the convection.

- The modified particle-density increment $N_B$ has a negligible effect, because this parameter always appears only in the perturbed energy equation (22) as a product with the inverse of the Lewis number $L_e$ near the temperature gradient and the volume fraction gradient of nanoparticles, such that:

$$N_B \sim 10^{-3} - 10^{-1} ; \; L_e \sim 10^2 - 10^3$$

- The presence of the nanoparticles in a base fluid allows us to destabilize it, this result can be interpreted as an increase in the volume fraction of nanoparticles, increases the Brownian motion and the thermophoresis of the nanoparticles, which cause the destabilizing effect.

- The regular fluids are more stable than the nanofluids.

- An increase in the temperature difference between the horizontal plates allows us to decrease the critical thermal Rayleigh number $R_{ac}$, this result can be explained by the increase in the buoyancy forces which destabilizes the system.

- To ensure the stability of the nanofluids, we can use the less dense nanoparticles or the ones which have a small heat capacity.

References


