International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2014): 5.611

Conditions of Boundedness and Compactness Sum of Operators on Weighted Berman Space

Eltigani Ismail Hassan[1]

[1]Al Imam Mohammad Ibn Saud Islamic University (IMSIU), College of Science, Department of Mathematics and Statistics

[2]University of Bahri & College of Applied Science and Industrial, Department of Mathematics

Abstract: ***In this paper, we study the product sum of operators which are bounded on the weighted Bergman space. We establish a necessary condition for the boundedness of the sum of Toeplitz products to be bounded or compact.***

Keywords: The weighted Bergman space, compact, Bounbedness

1. Introduction

The weighted Bergman space $A_\alpha^2(\mathbb{B}_n)$ is space of analytic functions on$\mathbb{B}_n$ which are square - integrable with respect to measure v_α on $\mathbb{B}_n$. Weighted Bergman spaces have studied by several authors in different context in [5],[6],[7] and [8]. In this article we consider the following necessary condition for sum of the Toeplitz products $T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}$ with $f_1,\bar{f}_2,\ldots,\bar{f}_r$ in $A_\alpha^2(\mathbb{B}_n)$,(See Theorem1 and Theorem2). Instead of products of two Toeplitz operators are bounded in $A_\alpha^2(\mathbb{B}_n)$ we use products of nth Toeplitz operators are bounded in weighted Bergman space . The normalized sum of reproducing kernel is given by $\sum_{r=1}^n K_\omega^\alpha = \sum_{r=1}^n \frac{(1-|\omega|^2)^{(\frac{n+\alpha+1}{2})}}{(1-\langle z,\omega\rangle)^{n+\alpha+1}}$, for $z,\omega \in \mathbb{B}_n$.

Suppose $f_1$and $f_2f_3\ldots f_r$ are in $A_\alpha^2(\mathbb{B}_n)$. Consider operator $f_1\otimes\bar{f}_2\otimes\ldots\otimes\bar{f}_r$ on weighted Bergman space defined by $(f_1\otimes\bar{f}_2\otimes\ldots\otimes\bar{f}_r)U = \langle U, f_1 f_2f_3\ldots f_r\rangle_{\alpha,} f_1$ for $U\in A_\alpha^2(\mathbb{B}_n)$. It easily prove that $f_1\otimes\bar{f}_2\otimes\ldots\otimes\bar{f}_r$ is bounded on $A_\alpha^2(\mathbb{B}_n)$ and $\|f_1\otimes\bar{f}_2\otimes\ldots\otimes\bar{f}_r\|_{\alpha,2}$=$\|f_1\|_{\alpha,2}\|f_2f_3\ldots f_r\|_{\alpha,2}$

$$f\otimes g = \sum_{r=1}^{n}\sum_{j=0}^{n+1+[\propto]}(-1)^j\sum_{|\gamma|=j}\frac{\Gamma(j+1-\alpha)}{\gamma!\,\Gamma(n+2+\alpha-j)}T_{z_r^\gamma}(T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r})T_{\bar{z}_r^\gamma}$$

$$+(-1)^{n+[\propto]}\frac{\Gamma(n+2+\propto)\sin(\pi[\propto])}{\pi}$$

$$\times\sum_{r=1}^{n}\sum_{j=0}^{\infty}\sum_{|\gamma|=n+2+[\propto]+j}\frac{\Gamma(j+1-[\propto])}{\gamma!}T_{z_r^\gamma}(T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r})T_{\bar{z}_r^\gamma}$$

we see that there exists a finite constant $\bar{C}_\propto$ such that
$\|f_1\otimes f_2\otimes f_3\otimes\ldots\otimes f_r\| \le \bar{C}_\propto \|T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}\|$ (1)
thus $\|f_1\|_2\|f_2\|_2\|f_3\|_2\ldots.\|f_r\|_2 \le \bar{C}_\propto\|T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}\|$

It follows from $T_{f\circ\varphi_\omega}U_\omega^{*\alpha} = U_\omega^{*\alpha}T_f$, applied to f_1 *and* $\bar{f}_2,\ldots,\bar{f}_r$ .That $\sum_{r=1}^n T_{f_1\circ\varphi_\omega}T_{\bar{f}_2\circ\varphi_\omega}\ldots T_{\bar{f}_r\circ\varphi_\omega} = \sum_{r=1}^n(T_{f_1\circ\varphi_\omega}U_\omega^{*\alpha})U_\omega^{*\alpha}(T_{f_2\circ\varphi_\omega}U_\omega^{*\alpha})U_\omega^{*\alpha}\ldots(T_{f_r\circ\varphi_\omega}U_\omega^{*\alpha})U_\omega^{*\alpha} = \sum_{r=1}^n(U_\omega^{*\alpha}T_{f_1})U_\omega^{*\alpha}(U_\omega^{*\alpha}T_{\bar{f}_2})U_\omega^{*\alpha}\ldots(U_\omega^{*\alpha}T_{\bar{f}_r})U_\omega^{*\alpha}$ $=\sum_{r=1}^n U_\omega^{*\alpha}(T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r})U_\omega^{*\alpha}$ (2)

For all $\omega_r\,\epsilon\,A_\alpha^2(\mathbb{B}_n)$,so for $f_1 f_2f_3\ldots f_r\,\epsilon\,A_\alpha^2(\mathbb{B}_n)$, a necessary condition for the sum of Toeplitz product $T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}$ to be bounded on $A_\alpha^2(\mathbb{B}_n)$ is

$$\sum_{r=1}^{n} sup_{\omega\epsilon\mathbb{B}_n} B_\alpha[|f_1|^2(\omega)B_\alpha[|\bar{f}_2|^2(\omega)\ldots B_\alpha[|\bar{f}_r|^2(\omega)<\infty$$

We will also make use of the following continuity condition of the Berezin transform: if $S_N\to S$ in operator norm, then $B_\alpha[S](\omega)$=$\lim_{N\to\infty}B_\alpha[S_N](\omega)$,For each $\omega\,\epsilon\,\mathbb{B}_n$. The above statement is an immediate consequence of the following inequality: $|B_\alpha[S](\omega)-B_\alpha[S_N](\omega)| \le \|S-S_N\|$.

Also H_{f_1}be bounded in the L^2 norm , that is, in order that there exists a constant C such that for all f_2 in H^∞ .

2. A Necessary Condition for the Sum of Boundedness of the Toeplitz Product

Theorem1.Let -1 $<\alpha<\infty$, and $f_1\ f_2f_3\ldots f_r$ in $A_\alpha^2(\mathbb{B}_n)$ .If $T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}$ is bounded on $A_\alpha^2(\mathbb{B}_n)$, then $\sum_{r=1}^n sup_{\omega\epsilon\mathbb{B}_n} B_\alpha[|f_1|^2(\omega)B_\alpha[|\bar{f}_2|^2(\omega)\ldots B_\alpha[|\bar{f}_r|^2(\omega)<\infty$

Proof. Suppose that $f_1\ f_2f_3\ldots f_r$ are analytic on $A_\alpha^2(\mathbb{B}_n)$ such that the densely defined Toeplitz product $T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}$ is bounded on $A_\alpha^2(\mathbb{B}_n)$. Using operator identity

Theorem2.**Let -1**$<\alpha<\infty$ **,and** $f_1,\bar{f}_2,\ldots,\bar{f}_n$ is be in $A_\alpha^2(\mathbb{B}_n)$. Then the sum of $T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}$ is compact on $A_\alpha^2(\mathbb{B}_n)$ then its sum of Berezin transform vanishes near the unit spheres .$\sum_{r=1}^n B_\propto[T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}](\omega)\to 0$ as $|\omega|\to 1^-$.We have seen that $\sum_{r=1}^n B_\propto[T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}](\omega) = \sum_{r=1}^n f_1(\omega)\,f_2\overline{(\omega)}\ldots f_r\overline{(\omega)}$

$\sum_{r=1}^n|f_1(\omega)\,f_2\overline{(\omega)}\ldots f_r\overline{(\omega)}| = \sum_{r=1}^n|B_\propto[T_{f_1}T_{\bar{f}_2}\ldots T_{\bar{f}_r}](\omega)|\to 0$ (3)
as $|\boldsymbol{\omega}|\to 1^-$, and it follows from the Maximum Modulus Principle that $f_1\ f_2f_3\ldots f_r\equiv 0$.

3. An Application to Toelitz Operators

Theorem3. **Let** $f_1 \epsilon H^{\infty}(\mathbb{B}_n, dv_\alpha)$ and $f_1\, f_2 f_3 \dots f_r \epsilon (\mathbb{B}_n, dv_\alpha)$.Then $T_{f_1} T^*_{\bar{f}_2}\, T^*_{\bar{f}_3} \dots T^*_{\bar{f}_r}$ is compact if and only if

$\sum_{r=1}^{n} \lim_{\omega\to\delta B_n} \|f_1 \circ \varphi_\omega\|_{\alpha,2} \|f_2 \circ \varphi_\omega f_3 \circ \varphi_\omega \dots f_n \circ \varphi_\omega - P(f1\circ\varphi\omega f2\circ\varphi\omega f3\circ\varphi\omega \dots fr\circ\varphi\omega\alpha,2 = 0$ (4)

suppose $T_{f_1} T^*_{\bar{f}_2}\, T^*_{\bar{f}_3} \dots T^*_{\bar{f}_r}$ is compact .

$\sum_{r=1}^{n} \|\mathcal{Y}_\omega (T_{f_1} T^*_{\bar{f}_2}\, T^*_{\bar{f}_3} \dots T^*_{\bar{f}_r})\| \to 0$ as $\omega \to \delta B_n$. We have

$\sum_{r=1}^{n} \mathcal{Y}_\omega (T_{f_1} T^*_{\bar{f}_2}\, T^*_{\bar{f}_3} \dots T^*_{\bar{f}_r})$

$= \sum_{r=1}^{n} \|(T_{f_1} K_\omega^{(\alpha)}) \otimes (T_{\bar{f}_2} K_\omega^{(\alpha)} T_{\bar{f}_3} K_\omega^{(\alpha)} \dots T_{\bar{f}_r} K_\omega^{(\alpha)}\|$

$= \sum_{r=1}^{n} \|T_{f_1} K_\omega^{(\alpha)}\|_{\alpha,2} \|T_{\bar{f}_2} K_\omega^{(\alpha)} T_{\bar{f}_3} K_\omega^{(\alpha)} \dots T_{\bar{f}_r} K_\omega^{(\alpha)}\|_{\alpha,2}$

$= \sum_{r=1}^{n} \|f_1 \varphi_\omega\|_{\alpha,2} \|f_2 \circ \varphi_\omega f_3 \circ \varphi_\omega \dots f_r \circ \varphi_\omega - P(f_1 \circ \varphi_\omega f_2 \circ \varphi\omega f3\circ\varphi\omega \dots fr\circ\varphi\omega\alpha,2$ (5)

Remark4.Let $P{:}L^2(\mathbb{C}, d\lambda_\alpha) \to F^2_\alpha$ denote the orthogonal projection. If $\psi, \varphi\, \epsilon L^2(\mathbb{C})$,we can define a linear operators $T_\psi T_\varphi$ on F^2_α by

$T_\psi f T_\varphi g$ = P($\psi f\ \varphi g$)= P(ψf)(φg). It is clear that $T_\psi T_\varphi$ are bounded and

$\|T_\psi T_\varphi\| \leq \|\psi\|_\infty \|\varphi\|_\infty$ .It is also easy to verify that

$(T_\psi T_\varphi)_z = U_z (T_\psi U_z)(T_\varphi U_z)\, U_z = T_{\psi o \varphi_z} T_{\varphi o \psi_z}$

for all $z\epsilon\mathbb{C}$. In particular,$((T_\psi)_z 1)((T_\varphi)_z 1)$= P($\psi o \varphi_z$)($\varphi o \psi_z$)

It follows that

$\left|(T_\psi)_z 1(\omega))\right| \left|(T_\varphi)_z 1(\omega))\right| \leq \left(\|\psi\|_\infty \int_{\mathbb{C}} |e^{\alpha\omega\bar{u}}| d\lambda_{\alpha(u)}\right) \left(\|\varphi\|_\infty \int_{\mathbb{C}} |e^{\alpha\omega\bar{u}}| d\lambda_{\alpha(u)}\right) = \|\psi\|_\infty \|\varphi\|_\infty \left|e^{\alpha\left|\frac{\omega}{2}\right|}\right|^4 = \|\psi\|_\infty \|\varphi\|_\infty\, e^{\frac{\alpha}{16}|\omega|^2}$ for all $\omega\ \epsilon\ \mathbb{C}$. This shows that

$$sup_{z\epsilon\mathbb{C}} \int_{\mathbb{C}} \left|(T_\psi)_z 1)\right|^P \left|(T_\varphi)_z 1)\right|^P d\lambda_\alpha < \infty$$

4. Boundedness in L^P,$P > 1$

One can also ask about the boundedness of H_f in the L^Pnorm .That is, ask for which f one has the following estimate, there exists a constant C such that

$\|f_1 f_2 - P(f_1 f_2)\|_P \leq C\|f_2\|_P,\ f_2 \in H^\infty$

As in the case $P = 2$ there is no loss of generality to suppose at the outset that $f_1 \in L^P$.

Theorem. Let $1 < P < \infty$ and let $f_1 \in L^P$.The following are equivalent.

(a) H_f is bounded in the L^P norm

(b) $\sup_{a\in D} dist\ L^P(fo\varphi_a, A^P) < \infty$

(c) $\sup_{z\in D} \inf\left\{\left(\frac{1}{|D(z)|}\right) \int_{D(z)} |f-k|^P\, dA : k \in A^P\right\} < \infty$

(d) $f = f_1 + f_2$ where $(1-|z|)\bar{\partial} f_2(z)$ is bounded

$sup_{z\in D} \frac{1}{|D(z)|} \int_{D(z)} |f_1|^P dA < \infty$ (6)

Proof. In place of U_φ we have the following isometrics on L^P.

$V_\varphi = |\dot{\varphi}|^{\frac{2}{P}} fo\varphi$, Then the V_φ are invertible for any automorphism φ of D $V_\varphi^{-1} = V_{\varphi-1}$.Now, with $= \varphi_a$,

$V_{\varphi_a} H_f V_{\varphi_a}^{-1}(1) = V_{\varphi_a} f V_{\varphi_a}(1) - V_{\varphi_a} P f V_{\varphi_a}(1) = fo\varphi_a - k_a$ (7)

Where $k_a = V_{\varphi_a} P f V_{\varphi_a}(1) \in A^P$. We need where is the fact that P is bounded in the L^Pnorm for all $1 < P < \infty$. since V_{φ_a} are isometries ,(7) implies that $dist_{L^P}(fo\varphi_a, A^P) \leq \|H_f(1)\|_P \leq \|H_f\|$. So (b) follows from (a).

Now assume (b) so that there is a constant C and analytic functions k_a with $\int |fo\varphi_a - k_a|^P dA \leq C\ , a \in D$.Apply V_{φ_a} to get $\int |f - k_a o\varphi_a|^P\, |\dot{\varphi}_a|^2 dA \leq C$, a ∈ D This gives (c) in the same way , via $|\dot{\varphi}_a|^2 \geq C\left(\frac{1}{|D(a)|}\right) \mathcal{X}_{D(a)}$. If assume (c) we obtain $f = f_1 + f_2$ except that L^P integrals appear everywhere in place of L^2 integrals. Thus (d) follows from (c). The L^P version of the boundedness for M_{f_1}. We estimate the L^P norm of $H_{f_1} g$ by the L^P norm of $f_1 g$ as in the $P = 2$ case, but using the fact (6) is just the requirement for $|f_1|^P dA$ to be a Carleson measure for A^P [9,10,11].We estimate the L^P and $L^{\grave{P}}$ is used and Hölder's inequality replaces Cauchy- Schwarz. We need to make use of the facts that $H_{f_2} g \in \left(A^{\grave{P}}\right)^\perp$ and $L^{\grave{P}} = A^{\grave{P}} \oplus |A^P|^\perp$, which follow easily from the boundedness of P in L^P norm.

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