A Study on Queuing System with Parallel Servers

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Abstract: ***Consider a system of two parallel queues where each customer must leave after service through a common gate G. Assume that service times at the two stations I and II are independent and identically distributed with density function $f(w)$ on $[0,\infty)$, and that exist service takes a fixed length of time $\gamma > 0$. Suppose further that a I-customer may be served at station I only if the previous I-customer has completed exist service. In this paper the total service time is calculated.***

Keywords: Queueing system, parallel servers ,Integral equation

1. Introduction

If we assume in addition that entry to the servers also takes place through gate G, and that entry can occur only at moments when no customer is undergoing exist service, we have two server analogue of the eight server queueing system described [7].1974 as representing the situation of an IBM 2314 direct access storage device. We may think of the customers as requests to access disk modules I and II.

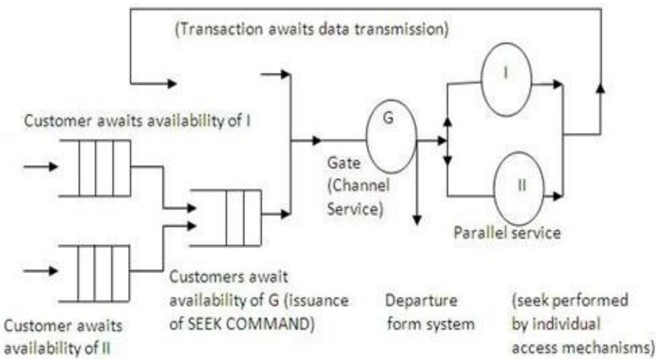

Figure 1.1

Schematic diagram for general system: service at G is instantaneous for incoming customers, and has positive constant duration for departing customers, in direct disk modules I and II. With the gate G representing the channel and exist service corresponding to data transmission [3]. The service at module I or II independently by individual access mechanisms. Thus in this situation, described schematically in figure 1.1. A I-customer arriving when the previous I-customer has left the system may still have to wait for entry it is happens that the current II-customers is in the exist process.

The waiting time W of a I-customer arriving at the queue is the time taken for station I to become ready to accept him, and the total service time is the sum $S_1 + W_G + \gamma$, where S_1 is the service time at station I, and W_G is the time spent waiting for exist service. Even if the arrivals are Poisson, the times W and W_G are not mutually independent, because the interference between the two queues at G forces both W and W_G to depend on the progress of the II-queue. However, in assessing the performance of the system, it appears at least for small γ to be a suitable approximation to regard the set-up as consisting of two parallel queues having independent Poisson arrivals, and having independent total service times with a suitable distribution [6]. Thus the determination of a distribution for $S_1 + W_G$ is of interest in itself.

At the time of entry of a I-customer to to station I, the distribution of $S_1 + W_G$, Namely his total service time less γ, depends both on the location of the II-customer currently being served (if any) and also on the number of II-customer waiting in the queue. However, the dependence on the size of the II-queue will be negligible if there are sufficiently many II-customer waiting that the possibility of this queue vanishing during the time $S_1 + W_G$ to be used in approximating the whole system.

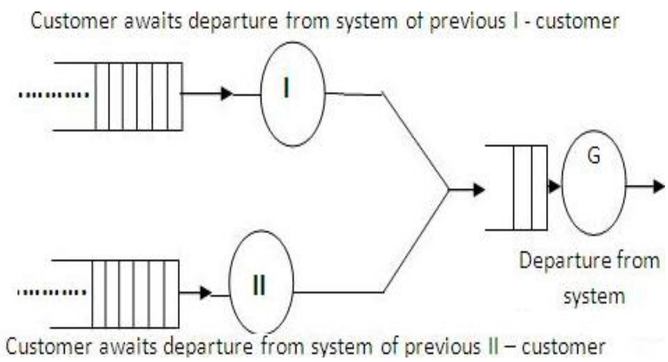

Figure 1.2: Schematic diagram for system with infinite queue size

Thus in the main part of this paper we shall make the simplifying that the queues for station I and II are inexhaustible, so that there is always one customer from each queue either in service, waiting for exist, or in the exist process. Since a I-customer [II-customer] will always enter as soon as the previous I-customer [II-customer] has left gate G, the simpler schematic diagram will suffice to represent the system.

2. The Embedded Markov Process

To begin with, let us consider the possible conditions which may obtain as a I-customer enter I-service. One possibility is that the II-customer currently in the system has been waiting for exist service at the time of departure of the previous I-customer. Thus as the current I-customer enters, the II-customer is beginning exist service at the gate G. Let us call this state of the system G_1 . The other possibility is that the current II-customer is in II-service, and has been for some length of time $s \geq \gamma$. Let us call this state of the system s_1.

Analogously, we may define states G_{II} and S_{II} which the system may occupy at the epochs of entry II-customers in to the system. Now if we define

X_n = State of the system at the epoch of entry of the nth customer after observation begins.

Its clear that $(X_n, n\geq 1)$ can be modeled as a Markov process on the state space.

$$X = [\gamma_I,\infty)\cup[\gamma_{II},\infty)\cup\{G_I\}\cup\{G_{II}\}$$

Where $[\gamma_I,\infty)=\{s_I : s\geq\gamma\}$ and $[\gamma_{II},\infty)$ are unconnected copies of the real interval $[\gamma,\infty)$ and $\{G_I\}$ and $\{G_{II}\}$ are isolated points. If $P(x,A)$ denotes the probability that $X_{n+1}\in A$ given $X_n = x$, Where $x\in X$ and $A\in B(X)$. the σ-algebra of Borel subsets of X, then defining

$$F(w)=\int_w^\infty f(r)\,dr \qquad (2.1)$$

We have

$$P(G_I,\{G_{II}\})=P(G_{II},\{G_I\})=1-F(\gamma)$$

$$P(G_I,\{\gamma_{II}\})=P(G_{II},\{\gamma_I\})=F(\gamma)$$

$$P(s_I,\{G_{II}\})=P(s_{II},\{G_{II}\})=\int_{v=0}^{\infty} f(v)\frac{F(s+v)-F(s+v+\gamma)}{F(s)}dv$$

and

$$P(s_I,\{G_{II}\})=P(s_{II},\{G_I\})=\int_{v=0}^{\infty} f(s+v)\frac{F(v)-F(v+\gamma)}{F(s)}dv$$

If $f(s)>0$. Moreover , each x in $[\gamma_I,\infty)\cup[\gamma_{II},\infty)$, the measure P(x,A) restricted to $[\gamma_I,\infty)$ or $[\gamma_{II},\infty)$ has a density P(x,y) with respected to lebesgue measure, and given by

$$P(s_I,v_{II})=$$

$$P(s_{II},v_I)=\int_{v=0}^{\infty} f(s+v-\gamma)\frac{F(v)}{F(s)} \quad \text{for } v\geq\gamma$$

$$=0 \qquad \text{for } v<\gamma \qquad (2.2)$$

Again provided that $f(s)>0$. (note that for negative arguments the service-time density f is 0).Now the Deoblin condition for the existence of an invariant probability measure for the process $(X_n, n\geq 1)$ (Doob [5] (1953)] will hold if it can be shown that

$$P(s_I,\{G_I\})+P(s_I,\{G_{II}\})$$

is bounded below as s varies. That is the following condition suffices.

Condition D.

$$\int_{v=0}^{\infty} f(v)\frac{F(s+v)-F(s+v+\gamma)}{F(s)}dv + \int_{v=0}^{\infty} f(s+v)\frac{F(v)-F(v+\gamma)}{F(s)}dv > \varepsilon$$

for some $\varepsilon>0$ and all t.Since condition D is easily seen to be satisfied for the particular cases of f treated . Let us assume hence forward that it is satisfied in the general formulation. For $(X_n, n\geq 1)$ there is only one ergodic set, i.e only one set s such that

$$P(x,S)=1 \qquad \text{if } x\in S$$

$$=0 \qquad \text{if } x\notin S$$

and this ergodic set is X. Also if $F(\gamma)>0$, as we shall also assume there are no cyclically moving sets i.e we cannot find disjoint $A_1, A_2,............A_r$ for $r\geq 2$ such that

$$P(x,A_j)=1 \text{ if } x\in A_{j-1}, j=2,....r$$

$$=1 \text{ if j=1 and } x\in A_r$$

it then follows (Doob (1953)) that there is a unique probability measure π on $(X, B(X))$ such that

$$\pi(A)=\int\pi(dx)\,P(x,A) \qquad (2.3)$$

and that for this measure π,

$$\lim_{m\to\infty}\| P_m(x,.)-\pi(.)\|=0$$

Where

$$P_m(x,A)=\Pr(X_{m+1}\in A)\,|\,X_1=x)$$

and $\|.\|$ denotes the total variation of the set function . Thus the in variant measure π may be interpreted as a steady-state distribution of X_n .

3. The Associated Semi-Markov Process

We may now define a process $\{X(t), t\in R+\}$ in real time talking values in X as follows. Let τ_n be the time of the nth transition epoch, and let

$$X(t) = X(\tau_n) = X_n \quad \text{if } \tau_n \le t < \tau_{n+1}$$
$$X(t) = X(0) \quad \text{if } 0 \le t < \tau_1$$

that is X(t) is the state attained at the most recent transition epoch prior to time t. The process $\{X(t), t \in R+\}$ may evidently be regarded as a semi-Markov process on (X, B(X)), the theory of semi-Markov process on general state spaces is discussed by clinar[4] (1969). Now let $T_n + \gamma$ be the time after τ_n of the first passage of the $\{X(t), t \in R+\}$ process into $[\gamma_1, \infty) \cup \{G_1\}$. If τ_n corresponds to the entry of I-customer, T_n may be regarded as the time to exist service of that customer or $S_1 + W_G$ in the notation of the introduction. let

$$G_w(w) = P(T_n \le w \mid X(\tau_n) = s_{II})$$
$$F_s(w) = P(T_n \le w \mid X(\tau_n) = s_I)$$
$$H(w) = P(T_n \le w \mid X(\tau_n) = G_I)$$

Then the quality of interest, namely the transform of the steady state distribution of $S_1 + W_G$, will be

$$\tilde{f}_+(\xi) = \int_0^\infty e^{i\xi w} df(w) \qquad (3.1)$$

where

$$\tilde{f}(w) = 2\int_{[\gamma_I,\infty)} F_{s_I}(w) d\pi(s_I) + 2\pi(\{G_I\}) H(w) \qquad (3.2)$$

and π denotes the steady-state distribution of X_n defined in 3.1. Since $\{X(t), t \in R+\}$ is a semi-markov on (X, B(X)) the following equations for first passage time probabilities are satisfied.

$$P(T_n \le w \mid X(\tau_n) = x)$$
$$= \int_{[\gamma_{II},\infty)\cup\{G_{II}\}} P(x,dy) \int_{u=0}^{w} d\Phi_{xy} P(T_{n+1} \le w-u \mid X(\tau_{n+1}) = y)$$
$$+ \int_{[\gamma_I,\infty)\cup\{G_{II}\}} P(x,dy) \Phi_{xy}(w-\gamma), \qquad (3.3)$$

where $\Phi_{xy}(u)$ is the conditional distribution of $\tau_{n+1} - \tau_n$, given that $X(\tau_n) = x, X(\tau_{n+1}) = y$.

Clearly , from this equation

$$H(w) = I(w-\gamma)(I - F(\gamma) + F(\gamma) G_\gamma(w-\gamma).$$

where

$$I(x) = 0, \quad x < 0$$
$$= 1, \quad x \ge 0$$

It is also easily seen under condition c to be described below , that G_s and F_s have densities $g_s(w)$ and $f_s(w)$ which are bounded uniformly in s for w in finite intervals. Then $f(w)$ for $w \ne \gamma$ will also have a density, given by

$$2\int_{[\gamma_I,\infty)} f_{s_I}(w) d\pi(s_I) + 2\pi(\{G_I\}) g_\gamma(w-\gamma). \qquad (3.4)$$

Moreover the densities $g_s(w)$ and $f_s(w)$ will satisfy a density version of (3.3), which will yield

$$F(s) g_s(w) =$$
$$F(w) f(s+w) + f(w-\gamma)(F(s+w-\gamma) - F(s+w))$$
$$+ \int_{r=0}^{w-\gamma} f(r+s) F(r+s+\gamma) g_{r+s+\gamma}(w-s-\gamma) dr \qquad (3.5)$$

and

$$F(s) f_s(w) =$$
$$F(s+w) f(w) + f(w+s-\gamma)(F(w-\gamma) - F(w) I(w-\gamma))$$
$$+ \int_{r=0}^{w-\gamma} f(r+s) F(r+\gamma) g_{r+\gamma}(w-r-\gamma) dr \qquad (3.6)$$

Thus

$$B(s,w) = F(s) g_s(w)$$

And

$$E(s,w) = F(s) f_s(w)$$

We have,

$$B(s,w) = A(s,w) + \int_{r=0}^{w-\gamma} f(r) B(r+s+\gamma, w-r-\gamma) dr \qquad (3.7)$$

And

$$E(s,w) = e(s,w) + \int_{r=0}^{w-\gamma} f(r+s) B(r+\gamma, w-r-\gamma) dr \qquad (3.8)$$

$$A(s,w) = F(w) f(s+w)$$
$$+ f(w-\gamma)(F(s+w-\gamma) - F(s+w)) \qquad (3.9)$$

And

$$e(s,w) = F(s+w) f(w)$$
$$+ (f(s+w-\gamma)(f(w-\gamma) - F(w)) I(w-\gamma) \qquad (3.10)$$

Thus if then (3.1), (3.2) and (3.4),

$$\tilde{f}_+(\xi) = 2\int_{u=0}^{\infty} D(u) \tilde{\varepsilon}_+(u+\gamma, \xi)$$
$$+ 2D_0 \tilde{\varepsilon}_+(\gamma, \xi) + 2C_0 e^{i\xi\gamma}[1 - f(\gamma) + \tilde{B}_+(\gamma, \xi)] \qquad (3.11)$$

Where

$$\tilde{\xi}_+(s,\xi) = \int_0^\infty e^{i\xi\gamma} \xi(s,w) dw$$

and

$$\tilde{B}_+(s,\xi) = \int_0^\infty e^{i\xi\gamma} B(s,w) dw$$

With a similar definition for $\tilde{B}_+(s,\xi)$ we have

$$\tilde{B}_+(s,\xi) = \tilde{e}_+(s,\xi)e^{i\xi\gamma}\int_{r=0}^{\infty} e^{i\xi\gamma} f(r+s)\tilde{B}_+(r+\gamma,\xi)dr \quad (3.12)$$

and thus to find $\tilde{f}_+(\xi)$ we need to determine $D_0, C_0, D(u)$ and $\tilde{B}_+(s,\xi)$ for $(s \geq \gamma)$ .It is possible to show that the densities $g_s(w)$ and $f_s(w)$ exist and are bounded uniformly in s for w in finite intervals if the following condition holds.

Condition C. The function

$$R(w) = \sup_s \frac{f(s+w)}{f(w)}$$

is also bounded for w in finite intervals .

First from this condition it follows that f(w) and

$S(w) = \sup_s \dfrac{f(s+w-\gamma) - f(s+w)}{f(s)}$ are bounded in finite intervals . Next it may be noted that for $w < \gamma, G_s(w), f_s(w)$ possess , namely

$$g_s(w)\frac{f(w)f(s+w)}{f(s)} \leq f(w)R(w) \quad \text{and}$$

$$f_s(w)\frac{f(s+w)f(w)}{f(s)} \leq f(w)$$

Finally , existence and boundedness of the densities for general w follows inductively with the aid of the probabilistic considerations used in establishing (3.5) and (3.6).

Condition C which again is easily seen to hold in the special cases of f treated in this paper, is also sufficient to establish the existence of the density D(u). Fro under C it is possible to show that $P(s_I, v_I)$ and $P(s_{II}, v_{II})$ are bounded uniformly in s for v in finite intervals. From this it follows that for any set Δ of lebesgue measure l(Δ) within a finite interval of (γ_I, ∞) $P_m(x,\infty) \leq Kl(\Delta)$ for some constant K and every m, and therefore that π restricted (γ_I, ∞) is absolutely continuous with respect to of lebesgue measure.

4. Conclusion

In this paper, we determined that the queues for station I and II are inexhaustible, so that there is always one customer from each queue either in service, waiting for exist, or in the exist process. Since a I-customer [II-customer] will always enter as soon as the previous I-customer [II-customer] has left gate G, the simpler schematic diagram will suffice to represent the system. At the time of entry of a I-customer to station I, the distribution of $S_1 + W_G$, Namely his total service time less γ, depends both on the location of the II-customer currently being served (if any) and also on the number of II-customer waiting in the queue. However, the dependence on the size of the II-queue will be negligible if there is sufficiently many II-customers waiting that the possibility of this queue vanishing during the time $S_1 + W_G$ to be used in approximating the whole system.

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