Lattice Points on the Homogeneous Cone $4(X^2+Y^2) - 3XY = 19Z^2$

Dr. P. Jayakumar[1], G. Shankarakalidoss[2]

[1]Department of Mathematics, A.V.V.M Sri Pushpam College (Autonomous), Poondi, Thanjavur-613503,Tamilnadu, India

[2]Department of Mathematics, Kings College of Engineering, Punalkulam, Pudukkottai (Dist) -613303, Tamilnadu, India

Abstract: ***The ternary quadratic homogeneous equation given by $4(X^2 + Y^2) - 3XY = 19Z^2$ is analyzed for its non-zero distinct integer points. Six different patterns of integer points satisfying under consideration are obtained. A few interesting relation between the solutions and special number patterns namely Polygonal number, Pyramidal number and Nasty number are presented.***

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1. Introduction

The ternary quadratic Diophantine equations offer an unlimited field of research due to their variety [1, 2]. For and extensive review of various problems, one may refer [3-6].This communication concerns with yet another interesting ternary quadratic equation $4(X^2 + Y^2) - 3XY = 19Z^2$ determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

Notations used

$T_{m,n}$ - Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solutions is

$$4(X^2 + Y^2) - 3XY = 19Z^2 \quad (1)$$

The Substitution of the linear transformations

$$X = u + v,\ Y = u - v,\ (u \neq 0,\ v \neq 0) \quad (2)$$

in (1) leads to

$$u^2 + 15v^2 = 19Z^2 \quad (3)$$

$$\text{Assume } Z(a, b) = a^2 + 15b^2,\ (a, b \neq 0) \quad (4)$$

We illustrate below six different patterns of non-zero distinct integer solutions to (1).

2.1 Pattern 1: Write 19 as

$$19 = (2 + i\sqrt{15})(2 - i\sqrt{15}) \quad (5)$$

Substitute (4) and (5) in (3) and Employing the method of factorization, define

$$(u + i\sqrt{15}v) = (2 + i\sqrt{15})(a + i\sqrt{15}b)^2$$

Equating real and imaginary parts, we get

$u = 2a^2 - 30b^2 - 30ab$

$v = a^2 - 15b^2 + 4ab$

Substituting the above values of u and v in (2), the non-zero distinct integer values for X and Y satisfying (1) are given by

$$X = X(a, b) = 3a^2 - 45b^2 - 26ab \quad (6)$$

$$Y = Y(a, b) = a^2 - 15b^2 - 34ab \quad (7)$$

Thus (4),(6) and (7) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, A(A+1)) + 180T_{3,A}^2 + 3P_A^5 = 3T_{4,A}$
2. $Y(A(A+1), (A+2)) - 4T_{3,A}^2 + 240P_A^3 + T_{32,A} \equiv -20 \pmod{34}$
3. $X(A, 1) - 3Y(A, 1) + Z(A,1) - T_{4,A} \equiv 15 \pmod{76}$
4. $X(A, 2) - T_{8,A} \equiv -30 \pmod{50}$
5. $Y(A, 1) - T_{4,A} \equiv -15 \pmod{34}$
6. $Y(1, B) + T_{32,B} \equiv 1 \pmod{48}$
7. $X(1, B) + T_{92,B} \equiv 3 \pmod{70}$
8. $X(2B, 2) - T_{26,A} \equiv 87 \pmod{93}$
9. $X(A+1, A+1) - T_{26,A} \equiv -68 \pmod{243}$
10. $-\frac{1}{2}Y(A, A)$, a Nasty number

2.2 Pattern: 2

Instead of (5) write 19 as

$$19 = (-2 + i\sqrt{15})(-2 - i\sqrt{15}) \quad (8)$$

Following the procedure as presented in pattern:1, the corresponding values of X and Y obtained from (2) are

$$X = X(a, b) = -a^2 + 15b^2 - 34ab \quad (9)$$

$$Y = Y(a, b) = -3a^2 + 45b^2 - 26ab \quad (10)$$

Thus (4), (9) and (10) represent non-zero distinct integral solutions of(1)in two parameters.

Properties:

1. $X(A, A(A + 1)) + T_{4,A} + 68P_A^5 = 60T_{3,A}^2$
2. $3X(A, 1) - Y(A, 1) + Z(A, 1) - T_{4,A} \equiv 15 \pmod{76}$
3. $Y(A, 1) + T_{8,A} \equiv 17 \pmod{28}$
4. $X(1, B) - T_{32,B} \equiv -1 \pmod{20}$
5. $X(2, 2A) + T_{122,A} \equiv -4 \pmod{77}$
6. $Y(2, B) - T_{92,B} \equiv -4 \pmod{8}$
7. $X(2A, 2) + T_{10,A} \equiv 60 \pmod{139}$
8. $X(A + 1, A + 1) + T_{42,A} \equiv -20 \pmod{59}$
9. $X(A(A + 1),(A + 2)) + 4T_{3,A}^2 + 204P_A^3 + T_{32,A} \equiv 60 \pmod{74}$

10. $\frac{3}{2}$ Y(A, A), a Nasty number

2.3 Pattern: 3

Consider 19 as

$$19 = \frac{(13 + i3\sqrt{15})(13 - i3\sqrt{15})}{16} \quad (11)$$

For this choice, the corresponding values of X and Y obtained from (2) are represented

$$X = X(a, b) = 4a^2 - 60b^2 - 16ab \quad (12)$$

$$Y = Y(a, b) = \frac{1}{2}[5a^2 - 75b^2 - 58ab] \quad (13)$$

The choices a =2A; b =2B in (4),(12) & (13) lead to

$$X = X(a, b) = 16A^2 - 240B^2 - 16AB \quad (14)$$

$$Y = Y(a, b) = 10A^2 - 150B^2 - 116AB \quad (15)$$

$$Z = Z(a, b) = 4A^2 + 60B^2 \quad (16)$$

Which represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $Y(A, A(A + 1)) + 600\,T^2_{3,A} + 232\,P^5_A = 10T_{4,A}$
2. $X(A, 2) - T_{34,A} \equiv -56 \pmod{113}$
3. $Y(A(A + 1), (A + 2)) - 40\,T^2_{3,A} + 696\,P^3_A + T_{302,A} \equiv -600 \pmod{749}$
4. $X(A, 1) - T_{34,A} \equiv -44 \pmod{49}$
5. $Y(1, B) + T_{302,B} \equiv 10 \pmod{265}$
6. $Y(2, B) + T_{302,B} \equiv 40 \pmod{381}$
7. $Z(A, 1) - 4T_{4,A} \equiv 0 \pmod{60}$
8. $-\frac{1}{12}$ {X(A, A)},a Nasty number
9. $-\frac{1}{18}$ {Y(A,A) + Z(A, A)}, a Nasty number

2.4 Pattern: 4

Consider 19 as

$$19 = \frac{(-13 + i3\sqrt{15})(-13 - i3\sqrt{15})}{16} \quad (17)$$

Following the analysis as presented in pattern:3, the corresponding non-zero distinct integer solution of (1) are found to be

$$X = X(a, b) = -10A^2 + 150B^2 - 116AB \quad (18)$$

$$Y = Y(a, b) = -16A^2 + 240B^2 - 64AB \quad (19)$$

Thus (18), (19) and (16) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, A(A + 1)) + 10T_{4,A} + 232\,P^5_A = 600\,T^2_{3,A}$
2. $X(A(A+1),(A + 2)) + 40\,T^2_{3,A} - T_{302,A} + 96\,P^5_A$ $= 600 \pmod{749}$
3. $X(1, B) - T_{302,B} \equiv -10 \pmod{33}$
4. $X(2, B) - T_{302,B} \equiv -40 \pmod{83}$
5. $Y(1, B) - T_{482,B} \equiv -16 \pmod{175}$
6. $Z(A, 1) - 4T_{4,A} \equiv 0 \pmod{60}$
7. $X(A + 1, A + 1) - T_{50,A} \equiv 24 \pmod{71}$
8. $Y(2, B) - T_{482,B} \equiv -64 \pmod{111}$
9. X(A, A), a Nasty number
10. $\frac{1}{4}$ {Y(A,A) - Z(A, A)}, a Nasty number

2.5 Pattern:5

The ternary quadratic equation (3) can be written as

$$(u^2 - 4Z^2) = 15(Z^2 - v^2) \quad (20)$$

Factorizing (20) we have

$$(u + 2Z)(u - 2Z) = 15(Z + v)(Z - v), \quad (21)$$

which is equivalent to the system of double equation

$$(2B - 15A)Z + Bu - 15Av = 0 \quad (22)$$

$$(B + 2A)Z - Au - Bv = 0 \quad (23)$$

Applying the method of cross multiplication we get

$$Z = -15A^2 - B^2 \quad (24)$$

$$u = -30A^2 + 2B^2 - 30AB \quad (25)$$

$$v = 15A^2 - B^2 - 4AB \quad (26)$$

Substituting (25) and (26) in (2) we get,

$$X = X(A, B) = -15A^2 + B^2 - 34AB \quad (27)$$

$$Y = Y(A,B) = -45A^2 + 3B^2 - 26AB \quad (28)$$

Thus (24), (27) and (28) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, 1) + T_{32,A} \equiv 1 \pmod{48}$
2. $X(2, 2B) - T_{10,B} \equiv -60 \pmod{133}$
3. $Y(1, B) - T_{8,B} \equiv -21 \pmod{24}$
4. $Y(B + 2, B + 2) + T_{138,B} \equiv -272 \pmod{339}$
5. $X(A, 2) + T_{32,A} \equiv 4 \pmod{82}$
6. $Y(2, B) - T_{8,B} \equiv -30 \pmod{50}$
7. $X(A, 1) + Y(A, 1) - 3Z(A, 1) + T_{32,A} \equiv 7 \pmod{74}$
8. 6{X(A, A) - Y(A, A) + Z(A, A)}, a Nasty number.
9. $-\frac{1}{2}$ {X(A,A)}, a Nasty number

2.6 Pattern:6

Also (21) is equivalent to the following two equations

$$(-15A - 2B)Z + Bu - 15Av = 0 \quad (29)$$

$$(-2A + B)Z - Au - Bv = 0 \quad (30)$$

Repeating the process as in pattern:5, the corresponding non-zero distinct integer solutions of (1) are given by

$$u = 30A^2 - 2B^2 - 30AB \quad (31)$$

$$v = 15A^2 - B^2 + 4AB \quad (32)$$

Substituting (30) and (31) in (2), we get

$$X = X(A,B) = 9A^2 - 8AB - 5B^2 \quad (33)$$

$$Y = Y(A,B) = 3A^2 - 16AB - B \quad (34)$$

Thus (24), (33) and (34) represents non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A,1) - T_{92,A} \equiv -3 \pmod{18}$
2. $X(A, 2) - T_{92,A} \equiv -4 \pmod{8}$
3. $Y(1, B) + T_{4,B} \equiv 15 \pmod{34}$
4. $X(A + 1, A + 1) - T_{34,A} \equiv 16 \pmod{47}$
5. $Y(2A, 2) - T_{122,A} \equiv -4 \pmod{77}$
6. $Y(2, B) + T_{4,B} \equiv 60 \pmod{68}$
7. $X(A, 1) + Y(A, 1) + Z(A, 1) - T_{92,A} \equiv -5 \pmod{16}$
8. $\frac{3}{2}$ X(B, B), a Nasty number.

3. Remarkable Observations

Employing the solution (X, Y, Z) of (1) each of the following expression among the special polygonal and pyramidal numbers are observed.

1. $1/19\{4[(3P^3_{x-2}/T_{3,x-2})^2+(P^5_y/T_{3,y})^2]-3(3P^3_{x-2}/T_{3,x-2})$

$(P^5_y/T_{3,y})\}$ is a perfect square.

2. $4[(P^5_x/T_{3,x})^2+(3P^3_{y-2}/T_{3,y-2})^2]-9(P^5_x/T_{3,x})^2$

$(P^3_{y-2}/T_{3,y-2})^2=14(6P^4_{z-1}/T_{3,2(z-)})^2$

3. $4[(P^5_x/T_{3,x})^2+(3P^3_{y-2}/T_{3,y-2})^2]-9(P^5_x/T_{3,x})^2$

$(P^3_{y-2}/T_{3,y-2})^2\equiv 0 \pmod{19}$

4. Conclusion

In this paper we have presented six different patterns of non-zero distinct integer solutions of the homogeneous equations given by $4(X^2+Y^2)-3XY=19Z^2$. To conclude one may search for other patterns of solution and their corresponding properties.

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Author Profile

P. Jayakumar received the B. Sc , M.Sc degrees in Mathematics from University of Madras in 1980 and 1983 and the M.Phil.,Ph.D degrees in Mathematics from Bharathidasan University, Thiruchirappalli in 1988 and 2010.He is now working as Associate Professor of Mathematics, A.V.V.M Sri Pushpam College Poondi,(Autonomous),Thanjavur (District) – 613 503, Tamil Nadu, India.