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Curvature Properties of Special (α, β)- Finsler Metrics

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Abstract: ***In Finsler space we see special metrics such as Randers metric, Kropina metric and Matsumoto metric.,etc. Curvature properties of special Finsler metrics are of different types in that mainly locally dually flatness, projectively flatness, conformal change and projective change between two special Finsler metrics.,etc. In this paper we are going study projective curvature properties of Special*** $(\alpha,\beta)-\mathrm{metric}$. ***Finsler metrics arise from Information Geometry. Such metrics have special geometric properties and will play an important role in Finsler geometry.*** $(\alpha,\beta)-\mathrm{metrics}$ ***are defined as the sum of a Riemannian metric and*** $1-form$.

Keywords and Phrases: Finsler metric; Special Finsler metric; $(\alpha,\beta)-\mathrm{metric}$; Randers metric; Douglas metric; Douglas tensor; Berwald space; Geodesic; Spray coefficients; Projective invariant; Projectively related metric; Projective change; Conformal change; Locally Minkowski space.

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1. Introduction

- The curvature properties of Finsler spaces have been studied by many authors([2], [3], [5], [6], [7], [10]).
- An interesting result concerned with the theory of projective change was given by Rapscak's paper. He proved necessary and sufficient conditions for projective change.
- S. Bacso and M. Matsumoto [1] discussed the projective change between Finsler spaces with $(\alpha,\beta)-metric$.
- H. S. Park and Y. Lee [4] studied on projective changes between a Finsler space with $(\alpha,\beta)-metric$ and the associated Riemannian metric.
- Recently some results on a class of $(\alpha,\beta)-metrics$ with constant flag curvature have been studied by Ningwei Cui, Yi-Bing Shen [11], N. Cui and Z. Lin.

2. Preliminaries

The Berwald's curvature tensor, Cartan's first and second curvature tensors are respectively given by

$$H^i_{hjk}=\partial_k G^i_{hj}+G^m_{hj}G^i_{mk}-(j/k)\,,$$

$$S^i_{hjk}=C^i_{mj}C^m_{hk}-(j/k)\,,$$

$$P^i_{hjk}=\partial_k F^i_{hj}-\partial_j C^i_{hk}+F^r_{hj}C^i_{rk}-C^r_{hk}F^i_{rj}+\partial_k N^r_j C^i_{hr}\,,$$

$$R^i_{hjk}=\partial_k F^i_{hj}+F^m_{hj}F^i_{mk}-(j/k)+C^i_{mk}R^m_{hj}\quad(2.1)$$

where, h-torsion tensor field is given by $R^i_{jk}=\partial_k N^i_j-(j/k)$ and (j/k) means interchange of the indices j and k with subtraction. On the basis of the following definitions we study curvature properties of special$(\alpha,\beta)-\mathrm{Finsler\ metric}$.

2.1 Definition

Finsler geometry is just Riemannian geometry without the quadratic restriction.

2.2 Definition

$(\alpha,\beta)-\mathrm{metrics}$ are defined as the sum of a Riemannian metric and $1-form$. If $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is 1-form satisfying$\|\beta_x\|_\alpha<b_0\ \forall x\in M$, then $L=\phi(s), s=\beta/\alpha$, is called an (regular) $(\alpha,\beta)-metric$. In this case, the fundamental form of the metric tensor induced by L is positive definite.

2.3 Definition

A Finsler metric is a scalar field $L(x,y)$ which satisfies the following three conditions:

(i) It is defined and differentiable at any point of $TM^n\backslash\{0\}$,

(ii) It is positively homogeneous of first degree in y^i, that is, $L(x,\lambda y)=\lambda L(x,y)$, for any positive number λ,

(iii) It is regular, that is, $g_{ij}(x,y)=\frac{1}{2}\dot\partial_i\dot\partial_j L^2$, constitute the regular matrix g_{ij}, where $\dot\partial_i=\frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x,y)$ is called Finsler metric $F^n=(M^n,L)$.

2.4 Definition

Two Finsler metrics L and $\bar{L}$ are projectively related if and only if their spray coefficients have the relation

$$G^i=\bar{G}^i+P(y)y^i\quad(2.2)$$

2.5 Definition

A Finsler metric is projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation

$$G^i_{\alpha} = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i \quad (2.3)$$

2.6 Definition

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on a common underlying manifold M^n. If any geodesic on F^n is also a geodesic on $\bar{F}^n$ and the converse is true, then the change $L \to \bar{L}$ of the metric is called a projective change.

The relation between the geodesic coefficients G^i of L and geodesic coefficients G^i_{α} of α is given by

$$G^i = G^i_{\alpha} + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1} y^i\}, \quad (2.4)$$

where, $\Theta = \dfrac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')}$,

$Q = \dfrac{\phi'}{\phi - s\phi'}$,

$\Psi = \dfrac{1}{2}\dfrac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}$

2.7 Definition

For a given Finsler metric $L = L(x, y)$, the geodesics of L satisfy the following ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4} g^{ij}\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\}.$$

Let $\phi = \phi(s), |s| < b_0$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, (|s| \le b \le b_0). \quad (2.5)$$

If $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is 1-form satisfying $\|\beta_x\|_\alpha < b_0 \ \forall x \in M$, then $L = \phi(s), s = \beta/\alpha$, is called an (regular) $(\alpha, \beta) - metric$.

In this case, the fundamental form of the metric tensor induced by L is positive definite.

2.8 Definition

Let $D^i_{jkl} = \dfrac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \dfrac{1}{n+1}\dfrac{\partial G^m}{\partial y^m} y^i\right)$, where G^i are the spray coefficients of L. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i, \quad (2.6)$$

Where $H^i_{00} = H^i_{jk} y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i \quad (2.7)$$

and $T^m_{y^m} = Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]$ (2.8)

Lemma 2.1: If $\alpha^2 \equiv 0(mod\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i y^i$ as a factor, then the dimension n is equal to 2 and b^2 vanishes.

3. Projective Change Between Two Finsler Metrics

In this section, we find the projective relation between two $(\alpha, \beta) - metrics$, that is, Special $(\alpha, \beta) - metric$ $L = c_1 + c_2\beta + \dfrac{\beta^2}{\alpha}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a same underlying manifold M of dimension $n > 2$.

From (2.3), $L = c_1 + c_2\beta + \dfrac{\beta^2}{\alpha}$ is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < \dfrac{1}{2}$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$\boldsymbol{\theta} = \dfrac{c_1c_2 - 4c_2s + c_2s^2 - 4s^3}{c_1^2 + 2c_1b^2 + 2c_2b^2s + 2s^2(b^2 - c_1) - 3c_2s^3 - 3s^4}$,

$\boldsymbol{Q} = \dfrac{c_2 + 2s}{c_1 - s^2}$,

$$\boldsymbol{\Psi} = \frac{1}{(c_1 + 2b^2) - 3s^2} \quad (3.1)$$

Substituting (3.1) into (2.4), we get

$G^i =$

$$G^i_\alpha + \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}s^i_0 + \left\{\frac{-2\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}s_0 + r_{00}\right\}\left\{\frac{\alpha^2}{(c_1 + 2b^2)\alpha^2 - 3\beta^2}b^i + \frac{(c_1c_2\alpha^3 - 4c_2\alpha^2\beta + c_2\alpha^2\beta^2 - 4\alpha\beta^3)y^i}{(c_1^2 + 2c_1b^2)\alpha^4 + 2c_2b^2\alpha^3\beta + 2(b^2 - c_1)\alpha^2\beta^2 - 3c_2\alpha\beta^3 - 3\beta^4}\right\}, \quad (3.2)$$

From (2.3), $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a regular Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

The geodesic coefficients are given by (2.4) with Randers metric,

$\bar{\boldsymbol{\theta}} = \dfrac{1}{2(1+s)}$,

$\bar{\boldsymbol{Q}} = \boldsymbol{1}$,

$$\bar{\boldsymbol{\Psi}} = \boldsymbol{0} \quad (3.3)$$

First we prove the following lemma:

Lemma 3.1*: Let $L = c_1 + c_2\beta + \dfrac{\beta^2}{\alpha}$ and $\bar{L} = \bar{\alpha} + \bar{\beta}$ be two $(\alpha, \beta) - metrics$ on a manifold M with dimension $n \ge 3$.*

Then they have the same Douglas tensor if and only if both the metrics L and $\bar{L}$ are Douglas metrics.

Proof: First, we prove the sufficient condition. Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, that is, both L and $\bar{L}$ have same Douglas tensor. Next, we prove the necessary condition. If L and $\bar{L}$ have the same Douglas tensor, then (2.5) holds. Substituting (3.1) and (3.3) in to (2.5), we obtain

$$H^i_{00} = \frac{A^i\alpha^{10} + B^i\alpha^8 + C^i\alpha^6 + D^i\alpha^4 + E^i\alpha^2}{I\alpha^8 + J\alpha^6 + K\alpha^4 - J\alpha^2 + L} - \bar{\alpha}\bar{s}^i_0 \quad (\mathbf{3.4})$$

Where

$$A^i = (n+1)(1+2b^2)\{c_2(c_1+2b^2)s^i_0 - 2c_2 s_0 b^i\},$$

$$B^i = (n+1)\{(1+2b^2)[2(c_1+2b^2)\beta s^i_0 - 4s_0 b^i\beta - 3c_2\beta^2 s^i_0] - 2(2+b^2)[c_2(c_1+2b^2)s^i_0 - 2c_2 s_0 b^i] - 4c_2(1 + 2b^2)s_0(c_1+2b^2)\},$$

$$C^i = (n+1)\{3[c_2(c_1+2b^2)s^i_0 - 2c_2 s_0 b^i + 6(2+b^2)c_2 s^i_0]\beta^4 - [6s^i_0 + 4(\beta + b^2)(c_1 + 2b^2)s^i_0 - 4s_0 b^i]\beta^3 - 4[(2c_1 + b^2)(1+2b^2)s_0]\beta^2 + r_{00} b^i\},$$

$$D^i = (n+1)\{12(2+b^2)s^i_0 r_{00} b^i \beta^5 + 3[c_2(c_1+2b^2)s^i_0 - 2c_2 s_0 b^i] + 6s_0(1+2b^2)\beta^4 + 2(2+b^2)r_{00} b^i\beta^3\},$$

$$E^i = (n+1)\{3\beta^4 r_{00} b^i - 12 s^i_0 \beta^7\}$$

$$\lambda = \frac{1}{n+1} \quad (\mathbf{3.5})$$

and $I = (1+n)(1+2b^2)c_1(c_1+2b^2)$,

$J = 2[2c_1^2 + 2c_1 + 2(c_1+1)b^4 + (c_1^2 + 8c_1 + 2)b^2]\beta^2$.

$K = (n+1)[3c_1^2 + 14c_1 b^2 + 44b^2 + 4b^4]\beta^4$,

$L = -12(n+1)[c_1 + b^2 + 1]\beta^6$,

$M = 9\beta^4$ $(\mathbf{3.6})$

Then (3.4) equivalent to

$$A^i\alpha^{10} + B^i\alpha^8 + C^i\alpha^6 + D^i\alpha^4 + E^i\alpha^2 = (I\alpha^8 + J\alpha^6 + K\alpha^4 + L\alpha^2 + M)(H^i_{00} + \bar{\alpha}\bar{s}^i_0) \quad (\mathbf{3.7})$$

Replacing y^i by $-y^i$ in (3.7) yields

$$-A^i\alpha^{10} + B^i\alpha^8 - C^i\alpha^6 + D^i\alpha^4 - E^i\alpha^2 = (I\alpha^8 - J\alpha^6 + K\alpha^4 - L\alpha^2 + M)(H^i_{00} - \bar{\alpha}\bar{s}^i_0) \quad (\mathbf{3.8})$$

Subtracting (3.8) *from* (3.7), we obtain

$$A^i\alpha^{10} + C^i\alpha^6 + E^i\alpha^2 = \alpha^2(J\alpha^6 + J\alpha^4 + K\alpha^2 + L) + \bar{\alpha}\bar{s}^i_0 M \quad (\mathbf{3.9})$$

From (3.9), $M\bar{\alpha}\bar{s}^i_0$ has the factor α^2, that is, the term $M\bar{\alpha}\bar{s}^i_0 = 7\beta^6\bar{\alpha}\bar{s}^i_0$ has the factor α^2.

Here, we study two cases for Riemannian metric.

Now, we study two cases for Riemannian metric.

Case (i): If $\bar{\alpha} \neq \mu(x)\alpha$, then $M\bar{s}^i_0 = 7\beta^6\bar{s}^i_0$ has the factor α^2.

Note that β^2 has no factor α^2. Then the only possibility is that $\beta\bar{s}^i_0$ has the factor α^2.

Then for each i there exists a scalar function $\tau^i = \tau(x)$ such that $\beta\bar{s}^i_0 = \tau^i\alpha^2$ which is equivalent to $b_j\bar{s}^i_k + b_k\bar{s}^i_j = 2\tau^i\alpha_{jk}$.

When $n \geq 3$ and we assume that $\tau^i \neq 0$, then

$$2 \geq rank\left(b_j\bar{s}^i_k\right) + rank\left(b_k\bar{s}^i_j\right) \geq \left(b_j\bar{s}^i_k + b_k\bar{s}^i_j\right) = rank\left(2\tau^i\alpha_{jk}\right) \geq 3, \quad (\mathbf{3.10})$$

Which is impossible unless $\tau^i = 0$. Then $\beta\bar{s}^i_0 = 0$. Since $\beta \neq 0$, we have $\bar{s}^i_0 = 0$, which implies that $\bar{\beta}$ is closed.

Case (ii): If $\bar{\alpha} = \mu(x)\alpha$, then (3.9) reduces to

$$A^i\alpha^9 + C^i\alpha^5 + E^i\alpha = \mu(x)\bar{s}^i_0(J\alpha^6 + J\alpha^4 + K\alpha^2 + L) + \bar{\alpha}M$$

Which can be written as

$$\mu(x)M\bar{s}^i_0 = [A^i\alpha^9 + C^i\alpha^5 + E^i\alpha - \mu(x)\bar{s}^i_0(J\alpha^6 + J\alpha^4 + K\alpha^2 + L)]\alpha^2 \quad (\mathbf{3.11})$$

From (3.11), we can see that $\mu(x)M\bar{s}^i_0$ has the factor α^2, that is, $\mu(x)M\bar{s}^i_0 = 7\bar{s}^i_0\beta^6$ has the factor α^2. Note that $\mu(x) \neq 0\ \forall x \in M$ and β^2 has no factor α^2. The only possibility is that $\beta\bar{s}^i_0$ has the factor α^2. As the similar reason in case (i), we have $\bar{s}^i_0 = 0$, when $n \geq 3$, which indicates that $\bar{\beta}$ is closed.

Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Thus $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric. Since L is projectively related to $\bar{L}$, then both L and $\bar{L}$ are Douglas metrics.

Now, we prove the following theorem:

Theorem 3.1: *The Finsler metric* $L = c_1 + c_2\beta + \frac{\beta^2}{\alpha}$ *is projectively related to* $\bar{L} = \bar{\alpha} + \bar{\beta}$ *if and only if the following conditions are satisfied*

$G^i_\alpha = G^i_{\bar{\alpha}} + \theta y^i - \tau\alpha^2 b^i$,

$b_{i|j} = \tau[(-1+2b^2)a_{ij} - 3b_i b_j]$,

$$d\bar{\beta} = 0, \quad (\mathbf{3.12})$$

Where $b^i = a^{ij}b_j$, $b = \|\beta\|_\alpha$, $b_{i|j}$ *denote the coefficients of the covariant derivatives of* β *with respect to* α, $\tau = \tau(x)$ *is a scalar function and* $\theta = \theta_i y^i$ *is a 1-form on a manifold M with dimension* $n \geq 3$.

Proof: Let us prove the necessary condition. Since Douglas tensor is an invariant under projective changes between two Finsler metrics, if L is projectively related to $\bar{L}$, then they have the same Douglas tensor. According to lemma 3.1, we get both L and $\bar{L}$ are Douglas metrics.

$L = c_1 + c_2\beta + \frac{\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = \tau\left[(-1+2b^2)a_{ij} - 3b_i b_j\right], \quad (\mathbf{3.13})$$

For some scalar function $\tau = \tau(x)$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta = b_i y^i$ with respect to α. In this case, β is closed. Since β is closed, $s_{ij} = 0 \Longrightarrow b_{i|j} = b_{j|i}$. Thus, $s^i_0 = 0$ and $s_0 = 0$.

By using (3.13), we have $r_{00} = \tau[(-1+2b^2)\alpha^2 - 3\beta^2]$.

Substituting all these in (3.2) we obtain

$$G^i = G^i_\alpha - \tau\left[\frac{c_1c_2\alpha^3 - 4c_2\alpha^2\beta + c_2\alpha^2\beta^2 - 4\alpha\beta^3}{(c_1{}^2 + 2c_1b^2)\alpha^4 + 2c_2b^2\alpha^3\beta + 2(b^2 - c_1)\alpha^2\beta^2 - 3c_2\alpha\beta^3 - 3\beta^4}\right]y^i + \tau\alpha^2 b^i \quad (3.14)$$

Since L is projective to $\bar{L}$, this is a Randers change between L and $\bar{\alpha}$. Noticing that $\bar{\beta}$ is closed, then L is projectively related to $\bar{\alpha}$. Thus there is a scalar function $P = P(y)$ on $TM\backslash\{0\}$ such that

$$\left[P + \tau\left(\frac{c_1c_2\alpha^3 - 4c_2\alpha^2\beta + c_2\alpha^2\beta^2 - 4\alpha\beta^3}{(c_1{}^2 + 2c_1b^2)\alpha^4 + 2c_2b^2\alpha^3\beta + 2(b^2 - c_1)\alpha^2\beta^2 - 3c_2\alpha\beta^3 - 3\beta^4}\right)\right]y^i = G^i_\alpha - G^i_{\bar{\alpha}} + \tau\alpha^2 b^i. \quad (3.16)$$

Note that the RHS of the above equation is a quadratic form. Then there must be a one form $\theta = \theta_i y^i$ on M, such that

$$P + \tau\left(\frac{c_1c_2\alpha^3 - 4c_2\alpha^2\beta + c_2\alpha^2\beta^2 - 4\alpha\beta^3}{(c_1{}^2 + 2c_1b^2)\alpha^4 + 2c_2b^2\alpha^3\beta + 2(b^2 - c_1)\alpha^2\beta^2 - 3c_2\alpha\beta^3 - 3\beta^4}\right) = \theta$$

.

Thus, (3.16) becomes

$$G^i_\alpha = G^i_{\bar{\alpha}} + \theta y^i - \tau\alpha^2 b^i. \quad (3.17)$$

Equations (3.12) and (3.13) together with (3.17) complete the proof of the necessity.

Since $\bar{\beta}$ is closed, it suffices to prove that L is projectively related to $\bar{\alpha}$. Substituting (3.13) in to (3.2) yields (3.14).

From (3.14) and (3.17), we have

$$G^i_\alpha = G^i_{\bar{\alpha}}\left[\theta - \tau\left(\frac{c_1c_2\alpha^3 - 4c_2\alpha^2\beta + c_2\alpha^2\beta^2 - 4\alpha\beta^3}{(c_1{}^2 + 2c_1b^2)\alpha^4 + 2c_2b^2\alpha^3\beta + 2(b^2 - c_1)\alpha^2\beta^2 - 3c_2\alpha\beta^3 - 3\beta^4}\right)\right]y^i,$$

That is, L is projectively related to $\bar{\alpha}$.

From the above theorem, we get the following corollaries.

Corollary 3.1: *The Finsler metric $L = c_1 + c_2\beta + \frac{\beta^2}{\alpha}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha = G^i_{\bar{\alpha}} + \theta y^i - \tau\alpha^2 b^i,$$

Where $b^i = a^{ij}b_j, \tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a one form on a manifold M with dimension $n \geq 3$.

Further, we assume that the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}_i y^i$ is a one form with

In this, we assume that the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}_i y^i$ is a one form with $\bar{b}_i = constants$. Then (3.12) can be written as

$$G^i_\alpha = \theta y^i - \tau\alpha^2 b^i,$$

$$b_{i|j} = \tau[(-1 + 2b^2)a_{ij} - 3b_i b_j] \quad (3.18)$$

Thus, we state

Corollary 3.2: *The Finsler metric $L = c_1 + c_2\beta + \frac{\beta^2}{\alpha}$ is projectivley related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if (3.18) holds.*

$$G^i_\alpha = G^i_{\bar{\alpha}} + Py^i \quad (3.15)$$

From (3.14) and (3.15), we have

4. Curvature properties of two (α, β)- Finsler metrics

In this section, we study the curvature properties of two $(\alpha, \beta) - metrics$, that is, Special $(\alpha, \beta) - metric$ $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a same underlying manifold M of dimension $n > 2$.

The Berwald curvature tensor of a Finsler metric L is defined by $= B^i_{jkl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$, where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$ and G^i are the spray coefficients of L. The mean Berwald curvature tensor is defined by $E = E_{ij}dx^i \otimes dx^j$, where $E_{ij} = \frac{1}{2}B^m_{mij}$. A Finsler metric is said to be of *isotropic mean Berwald curvature* if $E_{ij} = \frac{n+1}{2}c(x)L_{y^i y^j}$, for some scalar function $c(x)$ on M.

In this section, we assume that $(\alpha, \beta) - metric$ $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ has some special curvature properties. Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is projectively related to L.

First, we assume that L has isotropic $S - curvature$, that is, $S = (n + 1)c(x)L$ for some scalar function $c(x)$ on M. The $(\alpha, \beta) - metric$, $L = \alpha + \epsilon\beta + k(\frac{\beta^2}{\alpha})$ of isotropic curvature has been characterized, where ϵ and k are non zero constants. We use the following theorem proved by N. Cui.

Theorem 4.1: *For the special form of $(\alpha, \beta) - metric$, $L = \alpha + \epsilon\beta + k(\frac{\beta^2}{\alpha})$, where ϵ, k are non zero constants, the following are equivalent:*

(a) L has isotropic S-curvature, that is, $S = (n + 1)c(x)L$ for some scalar function $c(x)$ on M.

(b) L has isotropic mean Berwald curvature.

(c) β is a Killing one form of constant length with respect to α. This is equivalent to $r_{00} = s_0 = 0$.

(d) L has vanished S-curvature, that is, S=0.

(e) L is a weak Berwald metric, that is, $E = 0$.

The above theorem is valid for $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ when we take $\epsilon = 1$ and $k = -1$. Then we prove the following.

Theorem 4.2: *Let $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ has isotropic S-curvature or isotropic mean Berwald curvature. Then the*

Finsler metric L is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if the following conditions hold:
(a) α is projectively related to $\bar{\alpha}$,
(b) β is parallel with respect to α, that is, $b_{i|j} = 0$,
(c) $\bar{\beta}$ is closed, that is, $d\bar{\beta} = 0$ where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α..

Proof: The sufficiency is obvious from theorem 3.1. For the necessity, from theorem 3.1, we have that if L is projectively related to $\bar{L}$, then $b_{i|j} = \tau[(-1+2b^2)a_{ij} - 3b_i b_j]$, for some scalar function $\tau = \tau(x)$. Contracting above equation with y^i and y^j yields

$$r_{00} = \tau[(-1+2b^2)\alpha^2 - 3\beta^2] \qquad (4.1)$$

By the theorem 3.1, If L has isotropic $S-curvature$ or equivalently isotropic mean Berwald curvature, then $r_{00} = 0$. If $\tau \neq 0$, then (3.2) gives

$$(-1+2b^2)\alpha^2 - 3\beta^2 = 0\,, \qquad (4.2)$$

which is equivalent to

$$(-1+2b^2)a_{ij} - 3b_i b_j = 0. \qquad (4.3)$$

Contracting the above equation with a^{ij} yields $-n + (2n-3)b^2 = 0$, which is impossible. Thus, $\tau = 0$. Substituting in to theorem (3.1), we complete the proof.

Conclusion

If Finsler metric $L = c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}$ has $S-curvature$ or isotropic mean Berwald curvature. L is projectively flat if and only if $G^i_\alpha = Py^i$, $b_{i|j} = 0$ holds true. Then the Finsler metric L is projectivley related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if α is projectively related to $\bar{\alpha}$, β is parallel with respect to α, that is, $b_{i|j} = 0$ and $\bar{\beta}$ is closed, that is, $d\bar{\beta} = 0$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α.

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