$$u_2 = f(x_2) + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \left\{ (3^{1-\alpha} - 1) K_{21} u_1 + K_{22} u_2 \right\} \quad (25)$$

And for $j = 2, \ldots, k.$

$$u_j = f(x_j) + + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \sum_{i=1}^{j-1} \{(2j-(2i-1))^{1-\alpha} - (2j-(2i+1))^{1-\alpha}\} K_{ji} u_i + K_{jj} u_j,$$

This lead to:

$$\begin{cases} \left(1 - \frac{1}{(1-\alpha)} \frac{1}{(2k)^{1-\alpha}} K_{11}\right) u_1 = f(x_1) \\ \left(-\frac{3^{1-\alpha}-1}{(1-\alpha)(2k)^{1-\alpha}} K_{21}\right) u_1 + \left(1 - \frac{1}{(1-\alpha)(2k)^{1-\alpha}} K_{22}\right) u_2 = f(x_2) \\ \forall j = 2, \ldots, k \\ -\frac{1}{(1-\alpha)(2k)^{1-\alpha}} \sum_{i=1}^{j-1} \{(2j-(2i-1))^{1-\alpha} - (2j-(2i+1))^{1-\alpha}\} K_{ji} u_i + \left(1 - \frac{1}{(1-\alpha)(2k)^{1-\alpha}} K_{jj}\right) u_j = f(x_j) \end{cases}$$

(26)

The system (26) is a lower triangular system of algebraic equations with $O(k^2)$ operation gives column vector $\mathbf{u}$, then a desired approximation $u_k(t)$ of $u(t)$ is obtained.

3.2 Logarithm Kernel

Following the procedure presented in the section 3.1, we consider here the logarithmic singularities, let us denote by $g(t) = \ln|x-t|, 0 < t < x$. Using Taylor formulae the expansion of the function g is given by:

$$\ln|x-t| = \ln|x| - \sum_{m=1}^{\infty} \frac{1}{nx^n} t^n$$

The same techniques applied to Abel Kernel lead to:

$$\mathbf{u}^t \mathbf{\Phi}(x) = f(x) + \int_0^x \mathbf{\Phi}^t(x) \mathbf{K} \mathbf{\Phi}(t) \left(\ln|x| - \sum_{m=1}^{\infty} \frac{1}{nx^n} t^n \right) dt\, \mathbf{u}$$

$$= f(x) + \mathbf{\Phi}^t(x) \mathbf{K} \ln|x| \int_0^x \mathbf{\Phi}(t) \mathbf{\Phi}^t(t) dt\, \mathbf{u} - \sum_{m=1}^{\infty} \frac{1}{nx^n} \mathbf{\Phi}^t(x) \mathbf{K} \int_0^x \mathbf{\Phi}(t) \mathbf{\Phi}^t(t) t^n dt\, \mathbf{u} \qquad (27)$$

Let us use the collocation points $x_i = \frac{i - \frac{1}{2}}{k},\ for\ i = 1, \ldots, k$. Now, the equation (27) is evaluated at the collocation points reads as:

$$\mathbf{u}^t \mathbf{\Phi}(x_i) = f(x_i) + \mathbf{\Phi}^t(x_i) \mathbf{K} \ln|x_i| \int_0^{x_i} \mathbf{\Phi}(t) \mathbf{\Phi}^t(t) dt\, \mathbf{u} - \sum_{m=1}^{\infty} \frac{1}{nx_i^n} \mathbf{\Phi}^t(x_i) \mathbf{K} \int_0^{x_i} \mathbf{\Phi}(t) \mathbf{\Phi}^t(t) t^n dt\, \mathbf{u}$$

A simple calculation on can get the following system:

$$\begin{cases} u_1 = f(x_1) + \dfrac{\ln|x_1|}{2k}\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} - \dfrac{1}{2k}\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} \\ u_2 = f(x_2) + \dfrac{\ln|x_2|}{2k}\mathbf{e}_2^t\mathbf{K}(2\mathbf{d}^1+\mathbf{d}^2)\mathbf{u} - \dfrac{1}{2k}\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)3^n}\mathbf{e}_2^t\mathbf{K}\left(2^{n+1}\mathbf{d}^1 + (3^{n+1}-2^{n+1})\mathbf{d}^2\right)\mathbf{u} \\ \forall \quad j=2,\ldots,k: u_j = f(x_j) + \dfrac{\ln|x_j|}{2k}\mathbf{e}_j^t\mathbf{K}(2\sum_{i=1}^{j-1}\mathbf{d}^i + \mathbf{d}^j)\mathbf{u} \\ -\dfrac{1}{2k}\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)(2j-1)^n}\mathbf{e}_j^t\mathbf{K}\left(\sum_{i=1}^{j-1}\left((2i)^{n+1}-(2i-2)^{n+1}\right)\mathbf{d}^i + ((2j-1)^{n+1}-(2j-2)^{n+1})\mathbf{d}^j\right)\mathbf{u}. \end{cases}$$

Using $\mathbf{e}_j^t\mathbf{K}\mathbf{d}^i\mathbf{u} = K_{ji}u_i$, we get the following system:

$$\begin{cases} u_1 = f(\frac{1}{2k}) + \dfrac{\ln\left(\frac{1}{2k}\right)}{2k}K_{11}u_1 - \dfrac{1}{2k}K_{11}u_1 \\ u_2 = f(\frac{3}{2k}) + \dfrac{\ln\left(\frac{3}{2k}\right)}{2k}(2K_{21}u_1 + K_{22}u_2) - \dfrac{1}{2k}\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)3^n}\left(2^{n+1}K_{21}u_1 + (3^{n+1}-2^{n+1})K_{22}u_2\right) \\ \forall \quad j=2,\ldots,k: u_j = f(\frac{2j-1}{2k}) + \dfrac{\ln\left(\frac{2j-1}{2k}\right)}{2k}(2\sum_{i=1}^{j-1}K_{ji}u_i + K_{jj}u_j) \\ -\dfrac{1}{2k}\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)(2j-1)^n}\left(\sum_{i=1}^{j-1}\left((2i)^{n+1}-(2i-2)^{n+1}\right)K_{ji}u_i + ((2j-1)^{n+1}-(2j-2)^{n+1})K_{jj}u_j\right). \end{cases} \quad (28)$$

Using

$$\sum_{n=1}^{\infty}\frac{2^{n+1}}{n(n+1)3^n} = -\ln(3) + 2$$

$$\sum_{n=1}^{\infty}\frac{3^{n+1}-2^{n+1}}{n(n+1)3^n} = 1 + \ln(3)$$

$$\sum_{n=1}^{\infty}\frac{(2j-1)^{n+1}-(2j-2)^{n+1}}{n(n+1)(2j-1)^n} = 1 + \ln(2j-1), \forall j \geq 2$$

$$\sum_{n=1}^{\infty}\frac{(2i)^{n+1}-(2i-2)^{n+1}}{n(n+1)(2j-1)^n} = 2 + (2j-1-2i)\ln\left(\frac{2j-1-2i}{2j-1}\right)$$

$$-(2j-2i+1)\ln\left(\frac{2j-2i+1}{2j-1}\right), 1 \leq i \leq j-1$$

then we have a simple system:

$$\begin{cases} \left(1+\frac{1}{2k}\left(1-\ln\left(\frac{1}{2k}\right)\right)K_{11}\right)u_1 = f(\frac{1}{2k}) \\ \left(-\frac{1}{k}\ln\left(\frac{3}{2k}\right)+\frac{2-\ln(3)}{2k}\right)K_{21}u_1+\left(1-\frac{K_{22}}{2k}\ln\left(\frac{3}{2k}\right)+K_{22}\frac{1+\ln 3}{2k}\right)u_2 = f(\frac{3}{2k}) \\ \sum_{i=1}^{j-1}\eta_i u_i+\gamma_j u_j = f(\frac{2j-1}{2k}), \quad j=2,\ldots,k: \\ \eta_i = \left(-\frac{\ln\left(\frac{2j-1}{2k}\right)}{2k}+\frac{1}{2k}\left(2+(2j-1-2i)\ln\left(\frac{2j-1-2i}{2j-1}\right)-(2j-2i+1)\ln\left(\frac{2j-2i+1}{2j-1}\right)\right)\right)K_{ji} \\ \gamma_j = 1-\frac{K_{jj}}{2k}\ln\left(\frac{2j-1}{2k}\right)+K_{jj}\frac{1+\ln(2j-1)}{2k} \end{cases} \quad (29)$$

The system (29) is a lower triangular system of algebraic equations with $O(k^2)$ operation gives column vector $\mathbf{u}$, then a desired approximation $u_k(t)$ of $u(t)$ is obtained.

4. Error Analysis

In this section we assume that $u(t)$ is a differentiable function with bounded first derivative on (0,1), that is,
$\exists M>0; \quad \forall t\in(0,1): \ |u'(t)|\le M.$

The representation error when $u(t)$ is represented in a series of BPFs over every subinterval $[\frac{i-1}{k},\frac{i}{k})$ is

$$e_i(t)=\varphi_i\varphi_i(t)-u(t)$$
$$=\varphi_i-u(t).$$

We may proceed as follows:

$$\|e_i\|^2=\int_{\frac{i-1}{k}}^{\frac{i}{k}}e_i^2(t)dt=\int_{\frac{i-1}{k}}^{\frac{i}{k}}(\varphi_i-u(t))^2dt$$

$$=(\frac{i}{k}-(\frac{i-1}{k}))(\varphi_i-u(t_1))^2=\frac{1}{k}(\varphi_i-u(t_1))^2, \quad (\frac{i-1}{k}\le t_1<\frac{i}{k}) \quad (30)$$

where we used mean value theorem. As before, if

$$u(t)=\sum_{i=1}^{\infty}\varphi_i\varphi_i(t),$$

the i-th fourier coefficient is given by $\varphi_i=k<u(t),\varphi_i(t)>$. Using the mean value theorem leads to:

$$\varphi_i=k<u(t),\varphi_i(t)>=k\int_{\frac{i-1}{k}}^{\frac{i}{k}}u(t)dt=k(\frac{i}{k}-(\frac{i-1}{k}))u(t_2)=u(t_2), \quad (\frac{i-1}{k}\le t_2<\frac{i}{k}) \quad (31)$$

Now, we obtain:

$$\|e_i\|^2=\frac{1}{k}(\varphi_i-u(t_1))^2=\frac{1}{k}(u(t_2)-u(t_1))^2$$

$$=\frac{1}{k}(t_2-t_1)^2u'^2(t_0) \quad (t_1<t_0<t_2)\le\frac{1}{k^3}M^2.$$

Now for $i<j$ we have $[\frac{i-1}{k},\frac{i}{k})\cap[\frac{j-1}{k},\frac{j}{k})=\varnothing$, so $\int_0^1 e_i(t)e_j(t)dt=0$. Therefore,

$$\|e\|^2=\int_0^1 e^2(t)dt=\int_0^1(\sum_{i=1}^{k}e_i(t))^2dt$$

$$= \int_0^1 \sum_{i=1}^{k} e_i^2(t)dt + 2\sum_{i<j}\int_0^1 e_i(t)e_j(t)dt = \sum_{i=1}^{k}\int_0^1 e_i^2(t)dt$$

$$= \sum_{i=1}^{k} \| e_i \|^2 \le \frac{1}{k^2} M^2,$$

Therefore, $\| e(t) \| = O(\frac{1}{k})$, where, $e(t) = u_k(t) - u(t)$.

5. Numerical Examples

Now for implementing the described method to solve weakly singular integral equation, we give 4 examples with exact solution for compare with approximate solution.

Example 1:

$$u(x) = x^6(1 - \frac{2048}{3003}\sqrt{x}) + \int_0^x \frac{u(t)}{\sqrt{x-t}} dt,$$

with exact solution $y(x) = x^6$.

Example 2

$$u(x) = \frac{1}{\sqrt{x+1}} - \pi + 2\arctan(\frac{1}{\sqrt{x}}) + \int_0^x \frac{u(t)}{\sqrt{x-t}} dt,$$

with exact solution $u(x) = \frac{1}{\sqrt{x+1}}$.

Example 3

$$u(x) = 4\sqrt{x} - 4\sqrt{x}\ln(2) + 2\sqrt{x}\ln(\frac{1}{x}) + \ln(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}} dt,$$

with exact solution $u(x) = \ln(x)$.

Example 4 :

$$u(x) = -x(\ln(x))^2 - 2x\ln(x) - 2x + \frac{1}{6}x\pi^2 + \int_0^x u(t)\ln(x-t)dt$$

with exact solution $u(x) = \ln(x)$. For the case of two singular kernel and exactly Abel and logarithmic kernel, let us verify numerically how the method presented behave. The convergence will be tested and the logarithmic convergence curve will be plotted. In the case of Abel kernel and taking $k = 64$, we can compute the error between the approximate solution u_{64} and the exact solution. The left Figure in Figure (1) shows the error $\log_{10}(\| e \|)$ where $\| e \| = | u(x) - u_{64}(x) |$ for the Abel problem. The right Figure in Figure (1) shows the error $\log_{10}(\| e \|)$ where $\| e \| = | u(x) - u_{64}(x) |$ for the logarithm problem. Let us notice in many cases we can not compute the exact solution with logarithmic kernel, so one can proceed as the following: we test the convergence using the difference of between two sum at upper lower index 2^m and 2^{m+1}, i.e. we define an error $e_m = u_{2^m}(x) - u_{2^{m+1}}(x)$, with different value of m Example 1 is solved in [8] using Bernstein polynomials and example 2 is solved in [10] using the application of transformations of Korobov, Laurie and Sidi type in combination with the trapezoidal quadrature rule, evidently, in both cases, the methods are somewhat more accurate than our method. However, in our method high order convergence can be obtained easily by increasing the value of parameter k (the number of Block Pulse Functions).

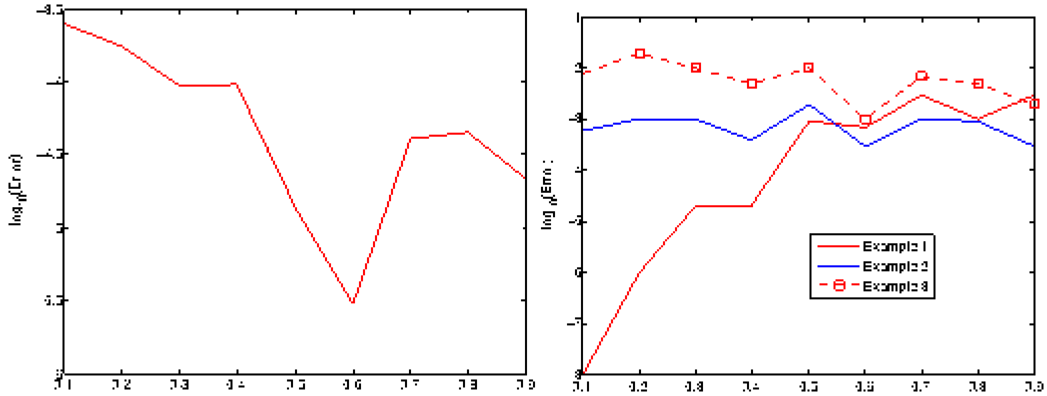

Figure 1: $\log_{10}(\|e\|)$ versus t. Using Abel Kernel (right figure) and Logarithm Kernel (left figure).

6. Conclusion

In present paper, a very simple and straight forward method, based on Block-Pulse Functions and Taylor expansion together with the collocation points is applied to solve the linear Volterra integral equations of the second kind with weakly singular kernel. For generalized orthogonal polynomials such as Legendre polynomials, Chebyshev polynomials and other, the calculation procedures are usually exhausting, see [1]. The advantages of presented method are low cost of setting up the equations and no use of any projection method. Also, the linear system of equations (25) is a lower triangular system which can be easily solved by forward substitution with $O(k^2)$ operations, therefore the count of operations is very low. Considering the results obtained in this paper, we plan in the future to tackle the following questions.

- Solve the Volterra-Fredholm integral equation with Abel and logarithm kernel using the same technique. Indeed, the Volterra-Fredholm integral can be transformed into a system of Fredholm equation, and this system can be solved using the present method.
- One can investigate a more general complicated problem, in higher dimension space.
- Solving a nonlinear Volterra integro-differential equations.
- Solving a nonlinear Volterra high-order integro-differential equation, it can be reduced to an algebraic system easy to solve.

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