A Computational Approach for Solving Singular Volterra Integral Equations with Abel Kernel and Logarithmic Singularities Using Block Pulse Functions

Sharefa EISA ALI ALHAZMI

Umm Al-Qura University College of Education for Girls at Al-Qunfudah , Mathematics department, Macca, KSA

Abstract: ***In this paper, a computational approach for solving singular Volterra integral equations with Abel kernel or logarithmic singularities will introduced. The technique of block-pulse functions will be used to solve the linear Volterra Integral equations with Abel kernel and Logarithm Kernels. This method is based on approximating of unknown function in terms of Block Pulse Functions and Taylor series expansion of singular part. The error analysis is presented to show the efficiency. Illustrative numerical examples are given to demonstrate the efficiency and simplicity of the proposed method in solving such types of systems of Abel or Logarithm integral equations.***

Keywords: Abel , logarithm kernel; Block-Pulse Function; Taylor expansion; Error analysis

1. Introduction

The Volterra integral equations are a special type of integral equations. These integral was studied by many author. These equations find applications in demography, the study of viscoelastic materials, and in insurance mathematics through renewal equation. Volterra integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. Several numerical methods for approximating the solution of Volterra integral equations with weakly singular kernel are known see for example [2, 4, 5, 6, 7, 8, 9, 10, 11]. For example, in [1] the author presented a method based on Legendre and Chebychev collocation method are presented to solve numerically the Fredholm Integral Equations with Abel kernel. The same technique can be used for the Volterra Integral equation. In the work of chniti ??, there are many complicated integral must be computed and difficult recurrence relation. I will try to avoid these difficult computation in this paper. In this paper we present a method based on the use of Taylor series expansion and Block Pulse Functions (BPFs) for solving a special case of singular Volterra integral equations of the second kind with logarithmic and Abel singularities defined as follows:

$$u(x) = f(x) + \int_0^x K(x,t)K_1(x,t)u(t)dt, \quad 0 \le t < x \le 1, \tag{1}$$

where functions f and K are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $u \in C[0,1]$ (see [3, 6]). The singular kernels will be investigate in this paper are

$$K_1(x,t) = \begin{cases} (x-t)^{-\alpha}, \quad 0 < \alpha < 1, \ \textit{for the Abel kernel} \\ \ln(|x-t|), \ \textit{for the Logarithm kernel} \end{cases}$$

for both kernels, we will present a strategy of solution using Block-Pulse Functions.

The paper is organized as follows. In section 2, we recall some property of the Block-Pulse Functions with exact proof. In section, 3, we present the system derived from Volterra Integral equation for both Abel and logarithm Kernel. In section, 4, we present the error analysis of our model problem. In section, 5, we present some numerical results to confirm the strategy proposed in our analysis.

2. Block-Pulse Functions

In this section, we define a k -set of Block-Pulse Functions (BPF) over the interval $[0,T[$ as:

$$\varphi_i(t) = \begin{cases} 1, & \dfrac{(i-1)T}{k} \le t < \dfrac{iT}{k}, \text{forall } i = 1,2,\dots,k \\ 0, & \text{elsewhere} \end{cases} \tag{2}$$

with a positive integer value for k. Also, φ_i is the i-th Block-Pulse Function. In this paper, it is assumed that T = 1, so BPFs are defined over $[0,1[$. BPFs, a set of orthogonal functions with piecewise constant values, are studied and applied extensively as a useful tool in the analysis, synthesis, identification and other problems of control and systems science. In comparison with other basis functions, BPFs can lead more easily to recursive computations to solve concrete problems. There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness. The disjointness property can be clearly obtained from the definition of BPFs:

$$\varphi_i(t)\varphi_j(t)=\begin{cases}0, & i\neq j\\ \varphi_i(t), & i=j\end{cases} \quad (3)$$

where, $i, j = 1,2,\ldots,k.$

The other property is orthogonality:

$$<\varphi_i(t),\varphi_j(t)>=\frac{1}{k}\delta_{ij} \quad (4)$$

where δ_{ij} is Kronecker delta.

The third property is completeness. To introduce the completeness we need the following result.

Proposition 2.1 *For every* $u\in C([0,1))$, *we have*

$$\lim_{k\to\infty}\| u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t)\|_{L^2[0,1)}=0$$

where

$$\alpha_{i,k}=k\int_0^1 u(t)\varphi_i(t)dt$$

Proof. First we have

$$\forall i\neq j,[\frac{i-1}{k},\frac{i}{k}[\cap[\frac{j-1}{k},\frac{j}{k}[=\varnothing.$$

Then

$$\int_0^1\varphi_i(t)\varphi_j(t)dt=0$$

$$\| u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t)\|^2=<u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t),u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t)>$$

$$=\int_0^1 u(t)^2dt+\sum_{i=1}^{k}\alpha_{i,k}^2\int_0^1\varphi_i^2(t)dt-2\sum_{i=1}^{k}\alpha_{i,k}\int_0^1 u(t)\varphi_i dt$$

$$+2\sum_{i\neq j=1}^{k}\alpha_{i,k}\alpha_{j,k}\int_0^1 B_j(t)\varphi_i(t)dt$$

$$=\int_0^1 u(t)^2dt+\sum_{i=1}^{k}\alpha_{i,k}^2\int_0^1\varphi_i(t)dt-2\sum_{i=1}^{k}\alpha_{i,k}\frac{\alpha_{i,k}}{k} \quad (5)$$

$$=\int_0^1 u(t)^2dt+\sum_{i=1}^{k}\frac{\alpha_{i,k}^2}{k}-2\sum_{i=1}^{k}\frac{\alpha_{i,k}^2}{k} \quad (6)$$

$$=\int_0^1 u(t)^2dt-\frac{1}{k}\sum_{i=1}^{k}\alpha_{i,k}^2 \quad (7)$$

Using, the mean value theorem we get:

$$\alpha_{i,k}^2=u(t_i)^2,\quad \frac{i-1}{k}\leq\square t_i\leq\frac{i\square}{k} \quad (8)$$

Substituting (8) into (7), we obtain:

$$\| u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t)\|^2=\int_0^1 u(t)^2dt-\frac{1}{k}\sum_{i=1}^{k}u(t_i)^2$$

From elementary calculus, we have:

$$\lim_{k\mapsto\infty}\| u(t)-\sum_{i=1}^{k}\alpha_{i,k}\varphi_i(t)\|^2=\lim_{k\mapsto\infty}\int_0^1 u(t)^2dt-\frac{1}{k}\sum_{i=1}^{k}u(t_i)^2=0$$

Proposition 2.2 *For every* $K\in C([0,1]\times[0,1])$, *we have*

$$\lim_{k\to\infty}\| K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\varphi_{j,k}(x)\varphi_{i,k}(t)\|_{L^2([0,1]\times[0,1])}=0$$

where

$$K_{i,j,k}=k^2\int_0^1\int_0^1 K(x,t)\varphi_{j,k}(x)\varphi_{i,k}(t)dxdt$$

Proof. Similarly to the proof of proposition 2.2. A simple calculus gives:

$$\| K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\varphi_{j,k}(x)\varphi_{i,k}(t)\|^2$$

$$=<K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}B_j(x)\varphi_i(t),K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}B_j(x)\varphi_i(t)> \quad (9)$$

$$=\int_0^1\int_0^1 K(x,t)^2dt+\sum_{j=1}^{k}\sum_{i=1}^{k}\int_0^1 K_{i,j,k}B_j^2(x)\varphi_i^2(t)dxdt$$

$$-2\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\int_0^1\int_0^1 K(x,t)\varphi_{j,k}(x)\varphi_{i,k}(t)dxdt$$

$$+2\sum_{(i,j)\neq(i',j')} K_{i,j,k}K_{i',j',k}\int_0^1\int_0^1 \varphi_{j,k}(x)\varphi_{i,k}(t)B_{j',k}(x)B_{i',k}(t)d\,xdt$$

$$=\int_0^1\int_0^1 K(x,t)^2\,d\,xdt+\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}^2\int_0^1\int_0^1\varphi_{j,k}(x)\varphi_{i,k}(t)d\,xdt-2\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\frac{K_{i,j,k}}{k}$$

$$=\int_0^1\int_0^1 K(x,t)^2\,d\,xdt+\sum_{j=1}^{k}\sum_{i=1}^{k}\frac{K_{i,j,k}^2}{k^2}-2\sum_{j=1}^{k}\sum_{i=1}^{k}\frac{K_{i,j,k}^2}{k^2}$$

$$=\int_0^1\int_0^1 K(x,t)^2\,d\,xdt-\frac{1}{k^2}\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}^2 \qquad (10)$$

Using, the mean value theorem we get:

$$K_{i,j,k}^2 = K(x_j,t_i)^2, \quad \frac{i-1}{k}\,\square\!\leq x_j,t_i \leq \frac{i\,\square}{k} \qquad (11)$$

Substituting (11) into (10), we obtain:

$$\|\,K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\varphi_{j,k}(x)\varphi_{i,k}(t)\,\|^2=\int_0^1\int_0^1 K(x,t)^2\,d\,xdt-\frac{1}{k^2}\sum_{i=1}^{k}\sum_{i=1}^{k}K(x_j,t_i)^2$$

From elementary calculus, we have:

$$\lim_{k\to\infty}\|\,K(x,t)-\sum_{j=1}^{k}\sum_{i=1}^{k}K_{i,j,k}\varphi_{j,k}\varphi_{i,k}(t)\,\|^2=\lim_{k\to\infty}\int_0^1\int_0^1 K^2(x,t)d\,xdt-\frac{1}{k^2}\sum_{j=1}^{k}\sum_{i=1}^{k}K(x_j,t_i)^2=0$$

Now we give the third property for BPFs, completeness. For every $u\in L^2[(0,1)^2)$, when k approaches to the infinity, Parseval's identity holds:

$$\int_0^1 u^2(t)dt=\lim_{k\to\infty}\sum_{i=1}^{k}u_{i,k}^2\,\|\,\varphi_{i,k}(t)\,\|^2,$$

where,

$$u_{i,k}=k<u(t),\varphi_i(t,k)>=k\int_0^1 u(t)\varphi_{i,k}(t)dt.$$

Now using all the results mentioned above we can expand functions $u(t)$ and $K(x,t)$ in terms of BPFs.

3. System derived from Volterra Integral equation

A function u defined over the interval $[0,1]$ may be expanded as:

$$u(t)=\sum_{i=1}^{\infty}u_i\varphi_{i,k}(t). \quad (12)$$

In matrix form

$$u(t)=\mathbf{u}^t\mathbf{\Phi}(t), \quad (13)$$

where,

- $\mathbf{u}=(u_i)_{i\in\mathrm{N}^*}$ is a infinite vector.
- $\mathbf{\Phi}(t)=(\varphi_{i,k})_{i\in\mathrm{N}^*}$ is a infinite vector.

Also, $K(x,t)\in L^2([0,1]^2)$ may be approximated as:

$$K(x,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}K_{ij,k}\varphi_{i,k}(x)\varphi_{j,k}(t),$$

or in matrix form

$$K(x,t)=\mathbf{\Phi}^t(x)\mathbf{K}\mathbf{\Phi}(t), \quad (14)$$

where

$$\mathbf{K}=\left(K_{ij,k}\right)_{1\leq i,j\in\mathrm{N}^*}, \quad K_{ij}=k^2\int_0^1\int_0^1 K(x,t)\varphi_{i,k}(x)\varphi_{j,k}(t)d\,xdt$$

Obviously, the infinite vector relation 13 and 14 must be truncated to used numerically. In particular, only a finite number of modes can be used to describe the solution. So, in a finite form one can write the same equation like in 13 and 14 but with $i, j = 1, \ldots, k$. The number of modes k to retain in the approximation must be fixed to get an accurate solution.

3.1 Abel Kenel

Now we define functions

$$g(t) = (x-t)^{-\alpha},$$

Taylor series expansion of $g(t)$ based on expansion about the point $t = 0$ leads to:

$$g(t) = \sum_{n=0}^{\infty}\prod_{m=0}^{n-1}(\alpha+m)x^{-\alpha-n}\frac{t^n}{n!}, \quad (15)$$

Substituting (15) into (1) we get

$$\mathbf{u}^t\boldsymbol{\Phi}(x) = f(x) + \sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)}{n!}x^{-\alpha-n}\boldsymbol{\Phi}^t(x)\mathbf{K}\int_0^x t^n\mathbf{D}(t)dt\,\mathbf{u}. \quad (16)$$

where the matrix $\mathbf{D}(t)$ is a diagonal matrix with diagonal coefficient $\beta_{ii} = \varphi_i(t)$. We set

$$\mathbf{h}(x) = \int_0^x t^n\mathbf{D}(t)dt, \quad (17)$$

For numerically reason, the infinite system 16 will be truncated and we set $i, j = 1, \ldots, k$, where k is the number of modes will be used to make our approximation. We define the nodes

$$x_j = \frac{j-\frac{1}{2}}{k}, \quad j = 1,2,\ldots,k$$

it's will be used to evaluate the function $\mathbf{h}(x)$. Now, we use the collocation points to evaluate the function $\mathbf{h}(x)$ at the nodes x_j for $j = 0,1,\ldots,k,$ we define the diagonal matrix matrix $\mathbf{d}^j$ as follows:

$$\mathbf{d}^j = \mathbf{D}(x_j), \quad \mathbf{d}^j_{mn} = \begin{cases}1, & m = n = j,\\ 0, & \text{elsewhere.}\end{cases} \quad (18)$$

- If $j = 1$ we have

$$\mathbf{h}(x_1) = \int_0^{\frac{1}{2k}} t^n\mathbf{D}(t)dt = \frac{1}{(n+1)(2k)^{n+1}}\mathbf{d}^1 \quad (19)$$

By evaluating the infinite system (16) after truncation at x_1 and using (19) we get

$$\begin{aligned} u_1 &= f(x_1) + \sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)}{n!}\left(\frac{1}{2k}\right)^{-\alpha-n}\mathbf{e}_1^t\mathbf{K}\frac{1}{(n+1)(2k)^{n+1}}\mathbf{d}^1\mathbf{u} \\ &= f(x_1) + \sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)}{(n+1)!}\frac{1}{(2k)^{1-\alpha}}\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} \\ &= f(x_1) + \frac{1}{(1-\alpha)}\frac{1}{(2k)^{1-\alpha}}K_{11}u_1, \quad (20)\end{aligned}$$

where $\mathbf{e}_1^t = (1,0,\ldots,0) \in \mathbb{R}^k$.
Using

$$\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} = K_{11}u_1, \quad \sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)}{(n+1)!} = \frac{1}{1-\alpha}$$

then

$$u_1 = f(x_1) + \frac{1}{(1-\alpha)} \frac{1}{(2k)^{1-\alpha}} K_{11} u_1. \quad (21)$$

- If $j = 2$ we have

$$\mathbf{h}(x_2) = \int_0^{\frac{3}{2k}} t^n \mathbf{D}(t)dt$$

$$= \int_0^{\frac{1}{k}} t^n \mathbf{d}^1 dt + \int_{\frac{1}{k}}^{\frac{3}{2k}} t^n \mathbf{d}^2 dt$$

$$= \frac{1}{(n+1)(2k)^{n+1}} \left\{ 2^{n+1}\mathbf{d}^1 + (3^{n+1} - 2^{n+1})\mathbf{d}^2 \right\}. \quad (22)$$

The infinite system (16) evaluate after truncation at x_2 and using (22) we get

$$u_2 = f(x_2) + \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(\alpha+m)}{n!} \left(\frac{3}{2k}\right)^{-\alpha-n} \mathbf{e}_2^t \mathbf{K} \frac{1}{(n+1)(2k)^{n+1}} \left\{ 2^{n+1}\mathbf{d}^1 + (3^{n+1} - 2^{n+1})\mathbf{d}^2 \right\} \mathbf{u}$$

$$= f(x_2) + \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(\alpha+m)}{(n+1)!} \frac{1}{(2k)^{1-\alpha}} 3^{-\alpha-n} \mathbf{e}_2^t \mathbf{K} \{ 2^{n+1}\mathbf{d}^1 + (3^{n+1} - 2^{n+1})\mathbf{d}^2 \} \mathbf{u},$$

therefore,

$$u_2 = f(x_2)$$

$$+ \frac{1}{(2k)^{1-\alpha}} \mathbf{e}_2^t \mathbf{K} \left\{ \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(\alpha+m) 3^{-\alpha-n} 2^{n+1}}{(n+1)!} \mathbf{d}^1 + \sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(\alpha+m) 3^{-\alpha-n} (3^{n+1} - 2^{n+1})}{(n+1)!} \mathbf{d}^2 \right\} \mathbf{u}$$

$$= f(x_2) + \frac{1}{(2k)^{1-\alpha}} \mathbf{e}_2^t \mathbf{K} \{ (\frac{1}{1-\alpha})(3^{1-\alpha} - 1)\mathbf{d}^1 + (\frac{1}{1-\alpha})\mathbf{d}^2 \} \mathbf{u}$$

$$= f(x_2) + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \mathbf{e}_2^t \mathbf{K} \left\{ (3^{1-\alpha} - 1)\mathbf{d}^1 + \mathbf{d}^2 \right\} \mathbf{u}$$

$$= f(x_2) + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \left\{ (3^{1-\alpha} - 1)K_{21}u_1 + K_{22}u_2 \right\},$$

Using

$$\mathbf{e}_2^t \mathbf{K} \{ (3^{1-\alpha} - 1)\mathbf{d}^1 + \mathbf{d}^2 \} \mathbf{u} = (3^{1-\alpha} - 1)K_{21}u_1 + K_{22}u_2,$$

$$\sum_{n=0}^{\infty} \frac{\prod_{m=0}^{n-1}(\alpha+m) 3^{-\alpha-n} (3^{n+1} - 2^{n+1})}{(n+1)!} = \frac{1}{1-\alpha}.$$

We conclude that

$$u_2 = f(x_2) + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \left\{ (3^{1-\alpha} - 1)K_{21}u_1 + K_{22}u_2 \right\}. \quad (23)$$

- In general, evaluating $\mathbf{h}(x)$ at the points $x_j (2 \le j \le k)$ leads to

$$\mathbf{h}(x_j) = \int_0^{\frac{j-\frac{1}{2}}{k}} t^n \mathbf{D}(t)dt$$

$$= \left(\sum_{i=1}^{j-1} \int_{\frac{i-1}{k}}^{\frac{i}{k}} t^n \mathbf{d}^i dt \right) + \int_{\frac{j-1}{k}}^{\frac{j-\frac{1}{2}}{k}} t^n \mathbf{d}^j dt$$

$$= \frac{1}{(n+1)(2k)^{n+1}}(\sum_{i=1}^{j-1}\{(2i)^{n+1}-(2i-2)^{n+1}\}\mathbf{d}^{i}+\{(2j-1)^{n+1}-(2j-2)^{n+1}\}\mathbf{d}^{j}), \quad (24)$$

The infinite system (16) after truncation evaluate at x_j and using (24) we get

$$u_j = f(x_j)+\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)}{n!}(\frac{2j-1}{2k})^{-\alpha-n}\mathbf{e}_j^t\mathbf{K}\frac{1}{(n+1)(2k)^{n+1}}\{\sum_{i=1}^{j-1}\{(2i)^{n+1}-(2i-2)^{n+1}\}\mathbf{d}^{i}$$
$$+ \quad \{(2j-1)^{n+1}-(2j-2)^{n+1}\}\mathbf{d}^{j}\}\mathbf{u}$$

$$= f(x_j)+\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)(2j-1)^{-\alpha-n}}{(n+1)!}\frac{1}{(2k)^{1-\alpha}}\mathbf{e}_j^t\mathbf{K}\{\sum_{i=1}^{j-1}\{(2i)^{n+1}-(2i-2)^{n+1}\}\mathbf{d}^{i}$$
$$+ \quad \{(2j-1)^{n+1}-(2j-2)^{n+1}\}\mathbf{d}^{j}\}\mathbf{u}$$

$$= f(x_j)+\frac{1}{(2k)^{1-\alpha}}\mathbf{e}_j^t\mathbf{K}\{\sum_{r=1}^{j-1}\{\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)(2j-1)^{-\alpha-n}}{(n+1)!}\{(2r)^{n+1}-(2r-2)^{n+1}\}\mathbf{d}^{r}\}$$
$$+\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)(2j-1)^{-\alpha-n}}{(n+1)!}\{(2j-1)^{n+1}-(2j-2)^{n+1}\}\mathbf{d}^{j}\}\mathbf{u},$$

therefore, by calculating the series in the last equation we obtain

$$u_j = f(x_j)+\frac{1}{(1-\alpha)(2k)^{1-\alpha}}\mathbf{e}_j^t\mathbf{K}\{\sum_{r=1}^{j-1}\{(2j-(2r-1))^{1-\alpha}-(2j-(2r+1))^{1-\alpha}\}\mathbf{d}^{r}+\mathbf{d}^{j}\}\mathbf{u}$$

this implies that

$$u_j = f(x_j)+\frac{1}{(1-\alpha)(2k)^{1-\alpha}}\{\sum_{i=1}^{j-1}\{(2j-(2i-1))^{1-\alpha}-(2j-(2i+1))^{1-\alpha}\}K_{ji}u_i+K_{jj}u_j\},$$

Using

$$\mathbf{e}_j^t\mathbf{K}\left\{\sum_{r=1}^{j-1}\{(2j-(2r-1))^{1-\alpha}-(2j-(2r+1))^{1-\alpha}\}\mathbf{d}^{r}+\mathbf{d}^{j}\right\}\mathbf{u} =$$

$$\sum_{i=1}^{j-1}\left\{(2j-(2i-1))^{1-\alpha}-(2j-(2i+1))^{1-\alpha}\right\}K_{ji}u_i+K_{jj}u_j,$$

and $\forall i = 2,\dots,j$ we have

$$\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)(2j-1)^{-\alpha-n}\{(2i-2)^{n+1}-(2i-4)^{n+1}\}}{(n+1)!}=\frac{(2j-(2i-3))^{1-\alpha}-(2j-(2i-1))^{1-\alpha}}{1-\alpha},$$

then

$$\sum_{n=0}^{\infty}\frac{\prod_{m=0}^{n-1}(\alpha+m)(2j-1)^{-\alpha-n}\{(2j-1)^{n+1}-(2j-2)^{n+1}\}}{(n+1)!}=\frac{1}{1-\alpha}.$$

We obtain for $j = 2,\dots,k.$

$$u_j = f(x_j)+\frac{1}{(1-\alpha)(2k)^{1-\alpha}}\sum_{i=1}^{j-1}\{(2j-(2i-1))^{1-\alpha}-(2j-(2i+1))^{1-\alpha}\}K_{ji}u_i+K_{jj}u_j$$

Finally, we will solve the following system:

$$u_1 = f(x_1)+\frac{1}{(1-\alpha)}\frac{1}{(2k)^{1-\alpha}}K_{11}u_1$$

$$u_2 = f(x_2) + \frac{1}{(1-\alpha)(2k)^{1-\alpha}}\left\{(3^{1-\alpha}-1)K_{21}u_1 + K_{22}u_2\right\} \quad (25)$$

And for $j = 2,\dots,k.$

$$u_j = f(x_j) + + \frac{1}{(1-\alpha)(2k)^{1-\alpha}} \sum_{i=1}^{j-1} \{(2j-(2i-1))^{1-\alpha} - (2j-(2i+1))^{1-\alpha}\} K_{ji}u_i + K_{jj}u_j ,$$

This lead to:

$$\left\{\begin{array}{r} \left(1-\frac{1}{(1-\alpha)}\frac{1}{(2k)^{1-\alpha}}K_{11}\right)u_1 = f(x_1) \\ \left(-\frac{3^{1-\alpha}-1}{(1-\alpha)(2k)^{1-\alpha}}K_{21}\right)u_1 + \left(1-\frac{1}{(1-\alpha)(2k)^{1-\alpha}}K_{22}\right)u_2 = f(x_2) \\ \forall j = 2,\dots,k \\ -\frac{1}{(1-\alpha)(2k)^{1-\alpha}}\sum_{i=1}^{j-1}\{(2j-(2i-1))^{1-\alpha} - (2j-(2i+1))^{1-\alpha}\}K_{ji}u_i + \left(1-\frac{1}{(1-\alpha)(2k)^{1-\alpha}}K_{jj}\right)u_j = f(x_j) \end{array}\right.$$

(26)

The system (26) is a lower triangular system of algebraic equations with $O(k^2)$ operation gives column vector $\mathbf{u}$, then a desired approximation $u_k(t)$ of $u(t)$ is obtained.

3.2 Logarithm Kernel

Following the procedure presented in the section 3.1, we consider here the logarithmic singularities, let us denote by $g(t) = \ln|x-t|, 0 < t < x$. Using Taylor formulae the expansion of the function g is given by:

$$\ln|x-t| = \ln|x| - \sum_{m=1}^{\infty} \frac{1}{nx^n} t^n$$

The same techniques applied to Abel Kernel lead to:

$$\mathbf{u}^t \mathbf{\Phi}(x) = f(x) + \int_0^x \mathbf{\Phi}^t(x)\mathbf{K}\mathbf{\Phi}(t)\left(\ln|x| - \sum_{m=1}^{\infty}\frac{1}{nx^n}t^n\right)dt\,\mathbf{u}$$

$$= f(x) + \mathbf{\Phi}^t(x)\mathbf{K}\ln|x| \int_0^x \mathbf{\Phi}(t)\mathbf{\Phi}^t(t)dt\,\mathbf{u} - \sum_{m=1}^{\infty}\frac{1}{nx^n}\mathbf{\Phi}^t(x)\mathbf{K}\int_0^x \mathbf{\Phi}(t)\mathbf{\Phi}^t(t)t^n dt\,\mathbf{u} \qquad (27)$$

Let us use the collocation points $x_i = \frac{i-\frac{1}{2}}{k},\ for\ i = 1,\dots,k$. Now, the equation (27) is evaluated at the collocation points reads as:

$$\mathbf{u}^t \mathbf{\Phi}(x_i) = f(x_i) + \mathbf{\Phi}^t(x_i)\mathbf{K}\ln|x_i| \int_0^{x_i} \mathbf{\Phi}(t)\mathbf{\Phi}^t(t)dt\,\mathbf{u} - \sum_{m=1}^{\infty}\frac{1}{nx_i^{\,n}}\mathbf{\Phi}^t(x_i)\mathbf{K}\int_0^{x_i}\mathbf{\Phi}(t)\mathbf{\Phi}^t(t)t^n dt\,\mathbf{u}$$

A simple calculation on can get the following system:

$$\begin{cases} u_1 = f(x_1) + \dfrac{\ln|x_1|}{2k}\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} - \dfrac{1}{2k}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}\mathbf{e}_1^t\mathbf{K}\mathbf{d}^1\mathbf{u} \\ u_2 = f(x_2) + \dfrac{\ln|x_2|}{2k}\mathbf{e}_2^t\mathbf{K}(2\mathbf{d}^1 + \mathbf{d}^2)\mathbf{u} - \dfrac{1}{2k}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)3^n}\mathbf{e}_2^t\mathbf{K}\left(2^{n+1}\mathbf{d}^1 + (3^{n+1} - 2^{n+1})\mathbf{d}^2\right)\mathbf{u} \\ \forall \quad j = 2,\ldots,k : u_j = f(x_j) + \dfrac{\ln|x_j|}{2k}\mathbf{e}_j^t\mathbf{K}(2\displaystyle\sum_{i=1}^{j-1}\mathbf{d}^i + \mathbf{d}^j)\mathbf{u} \\ -\dfrac{1}{2k}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)(2j-1)^n}\mathbf{e}_j^t\mathbf{K}\left(\sum_{i=1}^{j-1}\left((2i)^{n+1} - (2i-2)^{n+1}\right)\mathbf{d}^i + ((2j-1)^{n+1} - (2j-2)^{n+1})\mathbf{d}^j\right)\mathbf{u}. \end{cases}$$

Using $\mathbf{e}_j^t\mathbf{K}\mathbf{d}^i\mathbf{u} = K_{ji}u_i$, we get the following system:

$$\begin{cases} u_1 = f(\dfrac{1}{2k}) + \dfrac{\ln\left(\dfrac{1}{2k}\right)}{2k}K_{11}u_1 - \dfrac{1}{2k}K_{11}u_1 \\ u_2 = f(\dfrac{3}{2k}) + \dfrac{\ln\left(\dfrac{3}{2k}\right)}{2k}(2K_{21}u_1 + K_{22}u_2) - \dfrac{1}{2k}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)3^n}\left(2^{n+1}K_{21}u_1 + (3^{n+1} - 2^{n+1})K_{22}u_2\right) \\ \forall \quad j = 2,\ldots,k : u_j = f(\dfrac{2j-1}{2k}) + \dfrac{\ln\left(\dfrac{2j-1}{2k}\right)}{2k}(2\displaystyle\sum_{i=1}^{j-1}K_{ji}u_i + K_{jj}u_j) \\ -\dfrac{1}{2k}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)(2j-1)^n}\left(\sum_{i=1}^{j-1}\left((2i)^{n+1} - (2i-2)^{n+1}\right)K_{ji}u_i + ((2j-1)^{n+1} - (2j-2)^{n+1})K_{jj}u_j\right). \end{cases} \quad (28)$$

Using

$$\sum_{n=1}^{\infty}\frac{2^{n+1}}{n(n+1)3^n} = -\ln(3) + 2$$

$$\sum_{n=1}^{\infty}\frac{3^{n+1} - 2^{n+1}}{n(n+1)3^n} = 1 + \ln(3)$$

$$\sum_{n=1}^{\infty}\frac{(2j-1)^{n+1} - (2j-2)^{n+1}}{n(n+1)(2j-1)^n} = 1 + \ln(2j-1), \forall j \geq 2$$

$$\sum_{n=1}^{\infty}\frac{(2i)^{n+1} - (2i-2)^{n+1}}{n(n+1)(2j-1)^n} = 2 + (2j-1-2i)\ln\left(\frac{2j-1-2i}{2j-1}\right)$$

$$-(2j-2i+1)\ln\left(\frac{2j-2i+1}{2j-1}\right), 1 \leq i \leq j-1$$

then we have a simple system:

$$\left\{\begin{array}{r} \left(1+\frac{1}{2k}\left(1-\ln\left(\frac{1}{2k}\right)\right)K_{11}\right)u_1 = f(\frac{1}{2k}) \\ \left(-\frac{1}{k}\ln\left(\frac{3}{2k}\right)+\frac{2-\ln(3)}{2k}\right)K_{21}u_1+\left(1-\frac{K_{22}}{2k}\ln\left(\frac{3}{2k}\right)+K_{22}\frac{1+\ln 3}{2k}\right)u_2 = f(\frac{3}{2k}) \\ \sum_{i=1}^{j-1}\eta_i u_i+\gamma_j u_j = f(\frac{2j-1}{2k}), \quad j=2,\ldots,k: \\ \eta_i=\left(-\frac{\ln\left(\frac{2j-1}{2k}\right)}{2k}+\frac{1}{2k}\left(2+(2j-1-2i)\ln\left(\frac{2j-1-2i}{2j-1}\right)-(2j-2i+1)\ln\left(\frac{2j-2i+1}{2j-1}\right)\right)\right)K_{ji} \\ \gamma_j=1-\frac{K_{jj}}{2k}\ln\left(\frac{2j-1}{2k}\right)+K_{jj}\frac{1+\ln(2j-1)}{2k} \end{array}\right. \tag{29}$$

The system (29) is a lower triangular system of algebraic equations with $O(k^2)$ operation gives column vector $\mathbf{u}$, then a desired approximation $u_k(t)$ of $u(t)$ is obtained.

4. Error Analysis

In this section we assume that $u(t)$ is a differentiable function with bounded first derivative on (0,1), that is,
$\exists M>0;\quad \forall t\in(0,1):\quad |u'(t)|\leq M.$

The representation error when $u(t)$ is represented in a series of BPFs over every subinterval $[\frac{i-1}{k},\frac{i}{k})$ is

$$e_i(t)=\varphi_i\varphi_i(t)-u(t)$$
$$=\varphi_i-u(t).$$

We may proceed as follows:

$$\|e_i\|^2=\int_{\frac{i-1}{k}}^{\frac{i}{k}}e_i^2(t)dt=\int_{\frac{i-1}{k}}^{\frac{i}{k}}(\varphi_i-u(t))^2dt$$

$$=(\frac{i}{k}-(\frac{i-1}{k}))(\varphi_i-u(t_1))^2=\frac{1}{k}(\varphi_i-u(t_1))^2,\quad (\frac{i-1}{k}\leq t_1<\frac{i}{k}) \tag{30}$$

where we used mean value theorem. As before, if

$$u(t)=\sum_{i=1}^{\infty}\varphi_i\varphi_i(t),$$

the i-th fourier coefficient is given by $\varphi_i=k<u(t),\varphi_i(t)>$. Using the mean value theorem leads to:

$$\varphi_i=k<u(t),\varphi_i(t)>=k\int_{\frac{i-1}{k}}^{\frac{i}{k}}u(t)dt=k(\frac{i}{k}-(\frac{i-1}{k}))u(t_2)=u(t_2),\quad (\frac{i-1}{k}\leq t_2<\frac{i}{k}) \tag{31}$$

Now, we obtain:

$$\|e_i\|^2=\frac{1}{k}(\varphi_i-u(t_1))^2=\frac{1}{k}(u(t_2)-u(t_1))^2$$

$$=\frac{1}{k}(t_2-t_1)^2u^{'2}(t_0)\quad (t_1<t_0<t_2)\leq\frac{1}{k^3}M^2.$$

Now for $i<j$ we have $[\frac{i-1}{k},\frac{i}{k})\cap[\frac{j-1}{k},\frac{j}{k})=\varnothing$, so $\int_0^1 e_i(t)e_j(t)dt=0.$ Therefore,

$$\|e\|^2=\int_0^1e^2(t)dt=\int_0^1(\sum_{i=1}^{k}e_i(t))^2dt$$

$$= \int_0^1 \sum_{i=1}^{k} e_i^2(t)dt + 2\sum_{i<j}\int_0^1 e_i(t)e_j(t)dt = \sum_{i=1}^{k}\int_0^1 e_i^2(t)dt$$

$$= \sum_{i=1}^{k} \| e_i \|^2 \le \frac{1}{k^2} M^2,$$

Therefore, $\| e(t) \| = O(\frac{1}{k}),$ where, $e(t) = u_k(t) - u(t).$

5. Numerical Examples

Now for implementing the described method to solve weakly singular integral equation, we give 4 examples with exact solution for compare with approximate solution.

Example 1:

$$u(x) = x^6(1 - \frac{2048}{3003}\sqrt{x}) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt,$$

with exact solution $y(x) = x^6.$

Example 2

$$u(x) = \frac{1}{\sqrt{x+1}} - \pi + 2\arctan(\frac{1}{\sqrt{x}}) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt,$$

with exact solution $u(x) = \dfrac{1}{\sqrt{x+1}}.$

Example 3

$$u(x) = 4\sqrt{x} - 4\sqrt{x}\ln(2) + 2\sqrt{x}\ln(\frac{1}{x}) + \ln(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt,$$

with exact solution $u(x) = \ln(x).$

Example 4 :

$$u(x) = -x(\ln(x))^2 - 2x\ln(x) - 2x + \frac{1}{6}x\pi^2 + \int_0^x u(t)\ln(x-t)dt$$

with exact solution $u(x) = \ln(x).$ For the case of two singular kernel and exactly Abel and logarithmic kernel, let us verify numerically how the method presented behave. The convergence will be tested and the logarithmic convergence curve will be plotted. In the case of Abel kernel and taking $k = 64$, we can compute the error between the approximate solution u_{64} and the exact solution. The left Figure in Figure (1) shows the error $\log_{10}(\| e \|)$ where $\| e \| = | u(x) - u_{64}(x) |$ for the Abel problem. The right Figure in Figure (1) shows the error $\log_{10}(\| e \|)$ where $\| e \| = | u(x) - u_{64}(x) |$ for the logarithm problem. Let us notice in many cases we can not compute the exact solution with logarithmic kernel, so one can proceed as the following: we test the convergence using the difference of between two sum at upper lower index 2^m and 2^{m+1}, i.e. we define an error $e_m = u_{2^m}(x) - u_{2^{m+1}}(x)$, with different value of m Example 1 is solved in [8] using Bernstein polynomials and example 2 is solved in [10] using the application of transformations of Korobov, Laurie and Sidi type in combination with the trapezoidal quadrature rule, evidently, in both cases, the methods are somewhat more accurate than our method. However, in our method high order convergence can be obtained easily by increasing the value of parameter k (the number of Block Pulse Functions).

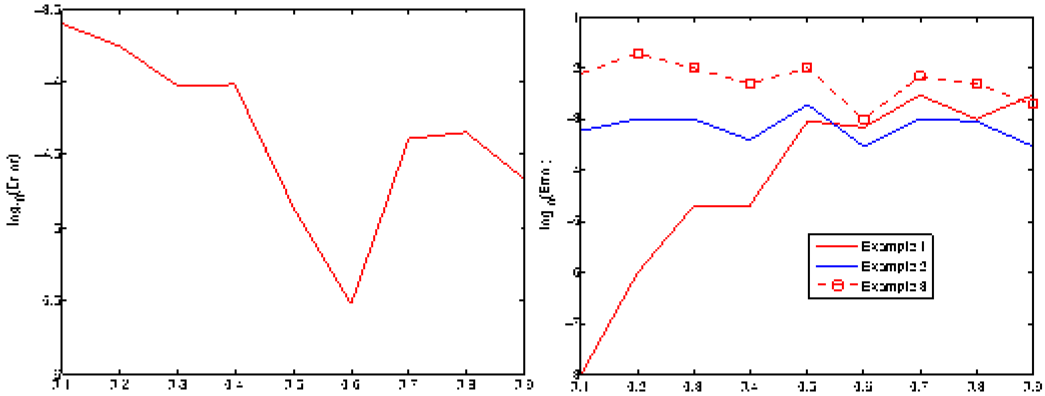

Figure 1: $\log_{10}(\| e \|)$ versus t. Using Abel Kernel (right figure) and Logarithm Kernel (left figure).

6. Conclusion

In present paper, a very simple and straight forward method, based on Block-Pulse Functions and Taylor expansion together with the collocation points is applied to solve the linear Volterra integral equations of the second kind with weakly singular kernel. For generalized orthogonal polynomials such as Legendre polynomials, Chebyshev polynomials and other, the calculation procedures are usually exhausting, see [1]. The advantages of presented method are low cost of setting up the equations and no use of any projection method. Also, the linear system of equations (25) is a lower triangular system which can be easily solved by forward substitution with $O(k^2)$ operations, therefore the count of operations is very low. Considering the results obtained in this paper, we plan in the future to tackle the following questions.

- Solve the Volterra-Fredholm integral equation with Abel and logarithm kernel using the same technique. Indeed, the Volterra-Fredholm integral can be transformed into a system of Fredholm equation, and this system can be solved using the present method.
- One can investigate a more general complicated problem, in higher dimension space.
- Solving a nonlinear Volterra integro-differential equations.
- Solving a nonlinear Volterra high-order integro-differential equation, it can be reduced to an algebraic system easy to solve.

References

[1] C.Chniti. Numerical Approximations of Fredholm Integral Equations with Abel Kernel using Legendre and Chebychev Polynomials. J. Math. Comput. Sci. 3 (2013), No. 2, 655-667.

[2] S. Abelman, D. Eyre, A rational basis for second kind Abel integral equations, Journal of Computational and Applied Mathematics, 34(1991)281-290.

[3] K. E. Atkinson, An existence theorem for Abel integral equations, SIAM J. Math. Anal. 5(1974)729-736. MR 51:3822

[4] E. Babolian, A. Salimi Shamloo, Numerical solution of Volterra integral and integro-differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions, Journal of Computational and Applied Mathematics, 214(2008)495-508.

[5] E. Babolian, Z. Masouri, Direct method to solve Volterra integral equation of the first kind using operational matrix with Block-Pulse Functions, Journal of Computational and Applied Mathematics, in press, 2007, doi: 10.1016/j.cam.2007.07.029.

[6] C. T. H. Baker, The numerical treatment of integral equations, Clarendon Press, Oxford, 1977. MR 57:7079

[7] P. Baratella, A. P. Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, J. Computat. Appl. Math. 163(2004)401-418.

[8] S. Bhattacharya, B. N. Mandal, Use of Bernstein Polynomials in Numerical Solutions of Volterra Integral Equations, Applied Mathematical Sciences, 2(2008)1773-1787.

[9] T. Diogo, P. Lima, Superconvergence of collocation methods for a class of weakly singular Volterra integral equations, Journal of Computational and Applied Mathematics, 218 (2008) 307-316.

[10] E. A. Galperin, E. J. Kansa, A. Makroglou, S. A. Nelson, Variable transformations in the numerical solution of second kind Volterra integral equations with continuous and weakly singular kernels; extensions to Fredholm integral equations, Journal of Computational and Applied Mathematics, 115 (2000) 193-211.

[11] S. A. Yousefi, Numerical solution of Abel integral equation by using Legendre wavelets, Appl. Math. Comput. 175(2006)574-580.